SUPPLEMENTAL MATERIALS: FERMIONIC BOUND STATES

To search for bound states of two Fermions with 3D spin-orbit coupling, we solve the two-particle Schrödinger's equation

$$[H_{so}(\mathbf{k}_1)\otimes\hat{1}+\hat{1}\otimes H_{so}(\mathbf{k}_2)]|\Psi\rangle+\hat{V}_{12}|\Psi\rangle=\Delta|\Psi\rangle, \quad (1)$$

where $\hat{V}_{12}$ is the two particle interaction potential, and the tensor product implies an operator the left operates on particle 1, and the operator on the right operates on particle 2. We assume the interaction to be purely local s-wave. This equation can be expressed in self-consistent form as

$$|\Psi\rangle=\hat{G}\hat{V}_{12}|\Psi\rangle \quad (2)$$

where the Green's function is defined as

$$\hat{G}=[(H_{so}-\Delta/2)\otimes\hat{1}+\hat{1}\otimes(H_{so}-\Delta/2)]^{-1}. \quad (3)$$

The ground state of a spin-orbit coupled atom will have energy $E_{so}=-mv^2/2$. We therefore search for solutions of (**??**) with energy $E<2E_{so}=-mv^2$. We define the binding energy as $\varepsilon=-mv^2-\Delta>0$.

The Green's function can be calculated by applying the unitary matrix $U=U_1\otimes U_2$, where

$$U_j=\exp\left[-i\frac{\theta_j}{2}\left(\mathbf{n}_j^*\cdot\hat{\boldsymbol{\sigma}}\right)\right] \quad (4)$$

is the unitary matrix that rotates from the original spin basis to the pseudo-spin basis defined by $|\alpha\beta\rangle$, with $\alpha,\beta=\pm1$, where a particle of spin $|\pm\rangle$ has energy in the $E=\mathbf{p}^2/2m\pm v|\mathbf{p}|$ band. The vector $\mathbf{n}_j^*=(-\sin\phi_j,\cos\phi_j,0)$, is perpendicular to both $\mathbf{k}_j=k_j(\cos\theta_j\cos\phi_j,\cos\theta_j\sin\theta_j,\sin\theta_j)$ and $\mathbf{e}_z$. The unitary matrix U_j transforms the Green's function to

$$\hat{G}=\sum_{\alpha\beta}\alpha\beta d_{\alpha\beta}U|\alpha\beta\rangle\langle\alpha\beta|U^\dagger \quad (5)$$

where $d_{\alpha\beta}=[s+v(\alpha k_1+\beta k_2)]^{-1}$.

We now assume the interaction potential is a short range, s-wave interaction, $\hat{V}_{12}=-v_0\delta(\mathbf{r}_1-\mathbf{r}_2)\mathcal{P}_s$, where $\mathcal{P}_s$ is a projector into the singlet state $|\psi_s\rangle=\frac{1}{2}(|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle)$. Using (**??**), we first apply the interaction potential to the state $|\Psi\rangle$ to get $V_{12}\langle\mathbf{k}|\Psi_B(\mathbf{Q})\rangle=\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\langle\psi_s,\mathbf{k}|\Psi_B(\mathbf{Q})\rangle|\psi_s\rangle$, where $\langle\mathbf{k}|\Psi_B(\mathbf{Q})\rangle$ is the momentum-space wavefunction of the relative coordinate $\mathbf{k}=(\mathbf{k}_1-\mathbf{k}_2)/2$. The center-of-mass momentum $\mathbf{Q}=\mathbf{k}_1+\mathbf{k}_2$ commutes with the Hamiltonain, and is thus a good quantum number which labels the bound state $|\Psi_B(\mathbf{Q})\rangle$. We can therefore express $|\Psi_B(\mathbf{Q})\rangle=N(\mathbf{Q})G(\mathbf{k},\mathbf{Q})|\psi_s\rangle$, where $N(\mathbf{Q})=\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\langle\psi_s,\mathbf{k}|\Psi_B(\mathbf{Q})\rangle$.

We then find the full wavefunction by applying the Green's function to the state $|\Psi(\mathbf{Q})\rangle$ to get

$$|\psi_B(\mathbf{Q})\rangle=N(\mathbf{Q})\sum_{\alpha\beta=\pm1}\alpha\beta d_{\alpha\beta}\chi_{\alpha\beta} \quad (6)$$

where $\alpha,\beta=\pm1$, and the spinors $\chi_{\alpha\beta}$ are most easily expressed in the coordinates $\mathbf{k}_1,\mathbf{k}_2$ as

$$\chi_{\alpha\beta}=\begin{pmatrix}(\beta+\cos\theta_1)\sin\theta_2e^{-i\phi_2}+(\alpha+\cos\theta_2)\sin\theta_1e^{-i\phi_1}\\(\alpha+\cos\theta_1)(-\beta+\cos\theta_2)-\sin\theta_1\sin\theta_2e^{i(\phi_2-\phi_1)}\\-(-\alpha+\cos\theta_1)(\beta+\cos\theta_2)+\sin\theta_1\sin\theta_2e^{-i(\phi_2-\phi_1)}\\(-\beta+\cos\theta_2)\sin\theta_1e^{i\phi_1}+(-\alpha+\cos\theta_1)\sin\theta_2e^{i\phi_2}\end{pmatrix} \quad (7)$$

in the basis of $(|\uparrow\uparrow\rangle,|\uparrow\downarrow\rangle,|\downarrow\uparrow\rangle,|\downarrow\downarrow\rangle)^T$. The $\chi_{\alpha\beta}$ form an orthogonal basis for each $\mathbf{k}_1,\mathbf{k}_2$, but are not normalized, since $\chi^\dagger_{\alpha'\beta'}\chi_{\alpha\beta}=\delta_{\alpha\alpha'}\delta_{\beta\beta'}c_{\alpha\beta}$, and $c_{\alpha\beta}=8\{1+\alpha\beta[\cos\theta_1\cos\theta_2+\cos(\phi_1-\phi_2)\sin\theta_1\sin\theta_2]\}$.
In this notation, the normalization is $N(\mathbf{Q})=\left(\int\frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^3}c_{\alpha\beta}d^2_{\alpha\beta}\right)^{-1/2}$.

The full wavefunction allows us to calculate the binding energy as follows. We substitute (**??**) into (**??**), and then integrate over the relative momentum coordinate. Due to symmetry, the triplet components of (**??**) will vanish under integration over the relative momentum, $\int\frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^3}\langle\mathbf{k}|\psi_B(\mathbf{Q})\rangle=\int\frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^3}\mathcal{P}_s\langle\mathbf{k}|\psi_B(\mathbf{Q})\rangle$. Therefore, left multiplying (**??**) by $\langle\mathbf{k}|\mathcal{P}_s$, and integrating over $\mathbf{k}$ allows us to express the self-consistency equation as

$$v_0\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\langle\psi_s|G(\mathbf{k},\mathbf{Q})|\psi_s\rangle=1. \quad (8)$$

The bound state $|\Psi_B(\mathbf{Q})\rangle$ will appear as a solution to this equation with energy $\Delta<-2\left(\frac{mv^2}{2}\right)$, which is twice the ground state of of a single spin-orbit coupled fermion. The integrals over the four modes $d_{\alpha\beta}$ have a linear ultraviolet divergence. To regularize the integrals we replace the interaction with a renormalized scattering length

$$\frac{1}{4\pi a_0}=\frac{1}{v_0}-\frac{1}{v_\infty} \quad (9)$$

where $1/v_\infty=\sum_{\alpha\beta}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{2m}{(k+\alpha mv)^2+(k+\beta mv)^2}$. The form of this regularizer is necessary to cure the linear divergence. A regularization scheme of $1/v_\infty\sim\int\frac{d^3\mathbf{k}}{(2\pi)^3}m/k^2$ will reduce the divergence from linear to logarithmic. Calculating the integrals using this regularization scheme above can be performed exactly, giving a binding energy

$$\varepsilon=\Delta-mv^2=-mv^2\left(mva_0\right)^2 \quad (10)$$

at $\mathbf{Q}=0$ to lowest order in ε/mv^2.