SUPPLEMENTAL MATERIALS: IMPLEMENTATION

The 3DSOC can be implemented in ^{87}Rb using two photon transitions. A possible implementation is given in Fig. ??. Nine laser beams with wavelength λ are used to couple states within the $F=1$ and $F=2$ hyperfine manifolds. A Zeeman field of $B=200$mT sets the quantization axis along the $\hat{z}$ direction. The remaining hyperfine transitions are isolated with a 6.8 GHz microwave field. Each pair of non-adjacent couplings is induced with three laser beams. All lasers have frequencies tuned between the D_1 and D_2 transitions. To ensure that all unwanted couplings are off resonance, we chose the base frequencies much larger than the splitting between the hyperfine levels $\omega_a, \omega_b, \omega_c \gg \delta_{ij}$.

The six optical couplings are induced in non-adjacent pairs on the state-linkage diagram. We list the properties of the three pairs of couplings independently. In what follows we define the vectors $\hat{e}_\pm = \mp\frac{1}{\sqrt{2}}(\hat{x} \pm \hat{y})$ and $\hbar k_L$ is the recoil momentum of the laser.

Couplings Ω_{12} and Ω_{34}

The two couplings Ω_{12} and Ω_{34} will be induced with three lasers, denoted L_a, L_{12} and L_{34}. The beam L_a will be shared in the two-photon couplings. The frequencies of the three beams will be chosen such that

$$\omega_{12} = \omega_a + \delta_{12} \quad (1)$$
$$\omega_{34} = \omega_a + \delta_{34}, \quad (2)$$

where ω_a is the frequency of the laser L_a and δ_{ij}, is the frequency splitting between the states $|i\rangle$ and $|j\rangle$. With such a configuration, the transition $|1\rangle \leftrightarrow |2\rangle$ and $|3\rangle \leftrightarrow |4\rangle$ will be on resonance, while all other dipole allowed transitions will be off resonance. The wavevectors of the lasers are given by

$$\boldsymbol{\kappa}_a = k_L\hat{z} \quad (3)$$
$$\boldsymbol{\kappa}_{12} = -k_L\hat{e}_- \quad (4)$$
$$\boldsymbol{\kappa}_{34} = k_L\hat{e}_-. \quad (5)$$

We can check that the effective momentum transfer of the couplings are given by

$$\mathbf{k}_{12} = k_L(\hat{e}_- + \hat{z}) \quad (6)$$
$$= \frac{k_L}{\sqrt{2}}\left(\hat{x} - \hat{y} + \sqrt{2}\hat{z}\right) \quad (7)$$
$$= \mathbf{K}_1 - \mathbf{K}_2 \quad (8)$$

and

$$\mathbf{k}_{34} = k_L(-\hat{e}_- + \hat{z}) \quad (9)$$
$$= \frac{k_L}{\sqrt{2}}\left(\hat{y} - \hat{x} + \sqrt{2}\hat{z}\right) \quad (10)$$
$$= \mathbf{K}_3 - \mathbf{K}_4. \quad (11)$$

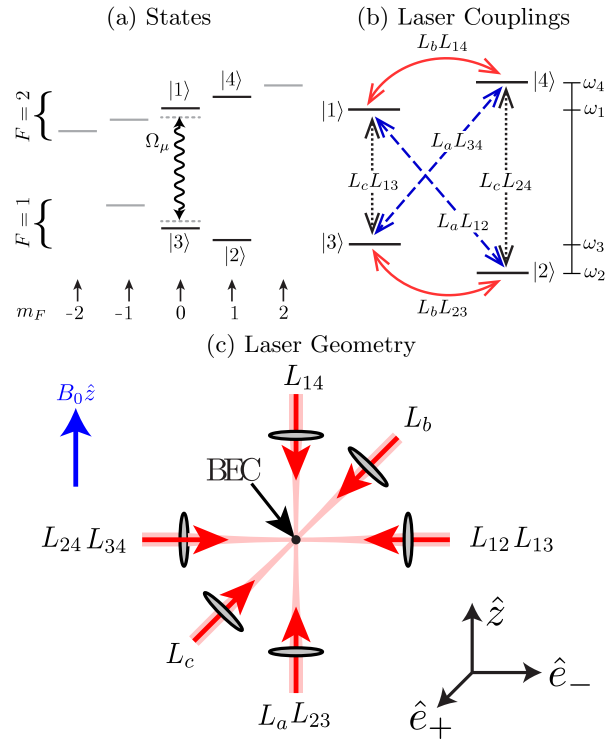

FIG. 1. Four hyperfine states $|F, m_F\rangle$ of ^{87}Rb are coupled using nine lasers. The quantization axis is set by a Zeeman field along the $\hat{z}$-axis. The couplings are produced in pairs. (a) The four states in the tetrahedral coupling are mapped to physical states according to $|1\rangle = |2,0\rangle$, $|2\rangle = |1,+1\rangle$, $|3\rangle = |1,0\rangle$, $|1\rangle = |2,+1\rangle$. (b) The frequencies of the three sets of lasers are given by $\{\omega_a, \omega_a+\delta_{12}, \omega_a+\delta_{34}\}$ (dashed blue), $\{\omega_b, \omega_b+\delta_{13}, \omega_b+\delta_{24}\}$ (dotted black), and $\{\omega_c, \omega_c+\delta_{14}, \omega_c+\delta_{23}\}$ (solid red), where $\delta_{ij} = \omega_i - \omega_j$ is the frequency difference between the states $|i\rangle$ and $|j\rangle$ in the rotating frame. (c) The geometry of the nine laser beams. $-\mathbf{k}_{12} = -\mathbf{k}_{13} = \mathbf{k}_{23} = \mathbf{k}_{34} = k_L\hat{e}_-$, $\mathbf{k}_a = \mathbf{k}_{23} = -\mathbf{k}_{14} = k_L\hat{z}$ and $\mathbf{k}_b = -\mathbf{k}_c = k_L\hat{e}_-$. The unit vectors $\hat{e}_\pm = \pm\frac{1}{\sqrt{2}}(\hat{x} \pm \hat{y})$.

Finally, the polarizations of the lasers will be chosen such that the beams L_{12} and L_{34} are linearly polarized along the $\hat{z}$ direction, while L_a is σ_+ polarized.

Couplings Ω_{14} and Ω_{23}

The couplings Ω_{14} and Ω_{23} will similarly be induced with three lasers, denoted L_b, L_{14} and L_{23}. The beam L_b will be shared in the two-photon couplings. The frequencies of the three beams will be chosen such that

$$\omega_{14} = \omega_b + \delta_{14} \quad (12)$$
$$\omega_{23} = \omega_b + \delta_{23}, \quad (13)$$

where ω_b is the frequency of the laser L_b. This choice of frequencies will isolate the transitions $|1\rangle \leftrightarrow |4\rangle$ and $|2\rangle \leftrightarrow |3\rangle$ in a manner similar to above couplings. The wavevectors of the lasers are given by

$$\boldsymbol{\kappa}_b = k_L \hat{e}_+ \tag{14}$$
$$\boldsymbol{\kappa}_{14} = -k_L \hat{z} \tag{15}$$
$$\boldsymbol{\kappa}_{23} = k_L \hat{z}. \tag{16}$$

We can check that the effective momentum transfer of the couplings are given by

$$\mathbf{k}_{14} = k_L(\hat{e}_+ + \hat{z}) \tag{17}$$
$$= \frac{k_L}{\sqrt{2}} \left(\hat{x} + \hat{y} + \sqrt{2}\hat{z} \right) \tag{18}$$
$$= \mathbf{K}_1 - \mathbf{K}_4 \tag{19}$$

and

$$\mathbf{k}_{23} = k_L(\hat{e}_+ - \hat{z}) \tag{20}$$
$$= \frac{k_L}{\sqrt{2}} \left(\hat{x} + \hat{y} - \sqrt{2}\hat{z} \right) \tag{21}$$
$$= \mathbf{K}_2 - \mathbf{K}_3. \tag{22}$$

Finally, the polarizations of the lasers will be chosen such that the beams L_b is linearly polarized along the $\hat{z}$ direction, while L_{14} and L_{23} are linearly polarized along the $\hat{x}$ axis.

Couplings Ω_{13} and Ω_{24}

The last pair of couplings Ω_{13} and Ω_{24} will similarly be induced with three lasers, denoted L_c, L_{13} and L_{24}. The beam L_c will be shared in the two-photon couplings. The frequencies of the three beams will be chosen such that

$$\omega_{13} = \omega_c + \delta_{13} \tag{23}$$
$$\omega_{24} = \omega_c + \delta_{24}, \tag{24}$$

where ω_c is the frequency of the laser L_c. This choice of frequencies will isolate the transitions $|1\rangle \leftrightarrow |3\rangle$ and $|2\rangle \leftrightarrow |4\rangle$ in a manner similar to above couplings. The wavevectors of the lasers are given by

$$\boldsymbol{\kappa}_c = -k_L \hat{e}_+ \tag{25}$$
$$\boldsymbol{\kappa}_{13} = -k_L \hat{e}_- \tag{26}$$
$$\boldsymbol{\kappa}_{24} = k_L \hat{e}_-. \tag{27}$$

We can check that the effective momentum transfer of the couplings are given by

$$\mathbf{k}_{13} = k_L(-\hat{e}_+ + \hat{e}_-) \tag{28}$$
$$= \frac{k_L}{\sqrt{2}} (2\hat{x}) \tag{29}$$
$$= \mathbf{K}_1 - \mathbf{K}_3 \tag{30}$$

and

$$\mathbf{k}_{24} = k_L(-\hat{e}_+ - \hat{e}_-) \tag{31}$$
$$= \frac{k_L}{\sqrt{2}} (2\hat{y}) \tag{32}$$
$$= \mathbf{K}_2 - \mathbf{K}_4. \tag{33}$$

Finally, the polarizations of the lasers will be chosen such that the beams L_{12} and L_{34} are linearly polarized along the $\hat{e}_+$ direction, while L_c is linearly polarized along $\hat{e}_-$.

Amplitude and Phase

In each pair of transitions, the three lasers provide a sufficient number of both amplitude and phase degrees of freedom to chose the the values of the couplings as desired in the main text.