Journal of Statistical Mechanics: Theory and Experiment
An IOP and SISSA journal

PAPER: Classical statistical mechanics, equilibrium and non-equilibrium

Coupled nonlinear stochastic differential equations generating arbitrary distributed observable with 1/*f* noise

J Ruseckas, R Kazakevičius and B Kaulakys

Institute of Theoretical Physics and Astronomy, Vilnius University, A. Goštauto 12, LT-01108 Vilnius, Lithuania
E-mail: julius.ruseckas@tfai.vu.lt

Received 14 January 2016
Accepted for publication 9 March 2016
Published 19 April 2016

Online at stacks.iop.org/JSTAT/2016/043209
doi:10.1088/1742-5468/2016/04/043209

Abstract. Nonlinear stochastic differential equations provide one of the mathematical models yielding $1/f$ noise. However, the drawback of a single equation as a source of $1/f$ noise is the necessity of power-law steady-state probability density of the signal. In this paper we generalize this model and propose a system of two coupled nonlinear stochastic differential equations. The equations are derived from the scaling properties necessary for the achievement of $1/f^{\beta}$ noise. The first equation describes the changes of the signal, whereas the second equation represents a fluctuating rate of change. The proposed coupled stochastic differential equations allow us to obtain a $1/f^{\beta}$ spectrum in a wide range of frequencies together with the almost arbitrary steady-state density of the signal.

Keywords: stochastic processes (theory), current fluctuations

J. Stat. Mech. (2016) 043209

1742-5468/16/043209+18$33.00

Contents

1. Introduction

Noise plays an essential role in many physical, biological and even social systems. Therefore, for the understanding of those systems it is important to characterize the noise and explain its origin. One of the characteristics used for description of noise is the power spectral density (PSD). In many cases the noise can be modeled as a white noise which has a frequency-independent PSD. However, there are various physical systems where noise has significant dependence on frequency. The characteristic behavior of the PSD is referred to as a 'color' of the noise. Pink noise or $1/f$ noise is a random process described by the PSD $S(f)$ inversely proportional to the frequency, that is $S(f) \propto 1/f^{\beta}$ with β close to 1. $1/f$ noise was observed first as an excess low-frequency noise in vacuum tubes [1, 2]. Later, such noise was found in condensed matter [3–7] and other systems [8–10]. Origin and the general nature of $1/f$ noise has been, up to now, the subject of a number of discussions and investigations: for reviews see [10–13].

Many models have been proposed to explain the origin of $1/f$ noise; for a short overview of the models see the introduction of [14]. In many condensed matter systems the $1/f$ spectrum is considered as a superposition of Lorentzians with a wide range distribution of relaxation times [5, 6, 15–18]. In this approach $1/f^{\beta}$ noise with the desirable slope β requires a certain distribution of parameters of the system [7, 8, 11, 17, 19, 20]. However, it has been shown that only several well-separated decay rates are sufficient to yield an approximately $1/f$ power spectrum [21]. Self-organized criticality (SOC) provides models of $1/f$ noise relevant for the understanding of driven non-equilibrium systems [22, 23]. The mechanism of SOC does not necessarily yield $1/f$ fluctuations [24, 25]. The $1/f$ noise in the fluctuations of a mass was first seen in a sandpile model with threshold dissipation proposed in [26] and was analytically obtained in a one-dimensional directed model of sandpiles [27]. Yet another model of $1/f$ noise represents the signals as sequences of the renewal pulses or events with the power-law distribution of the inter-event time [28]. Recently, thermal finite-size fluctuations as a mechanism for $1/f$ noise have been proposed [29].

In some systems the $1/f$ fluctuations are non-Gaussian [30, 31]. Power-law distribution of signal intensity as well as power-law behavior of the PSD in a wide range of frequencies can be obtained using point processes where the time between the adjacent pulses experience relatively slow the Brownian-like motion [32–34]. Starting from this point process model nonlinear stochastic differential equations (SDEs) generating $1/f^{\beta}$ noise have been derived in [14, 35, 36]. Such nonlinear SDEs have been applied to describe signals in socio-economical systems [37, 38] and as a model of neuronal firing [39].

However, in most cases $1/f$ noise is a Gaussian process [12, 40]. The drawback of the nonlinear SDEs generating signals with $1/f^{\beta}$ PSD, proposed in [35, 36], is the necessity of a power-law steady-state probability density function (PDF) of the signal. It is impossible to obtain a Gaussian PDF together with a $1/f$ spectrum from such nonlinear SDEs. The purpose of this paper is to remedy this drawback of nonlinear SDEs as a source of $1/f$ noise by considering not only one SDE, but a system of two coupled SDEs. In this system of coupled SDEs, we interpret the first equation as giving the signal, whereas the second equation represents a fluctuating rate of change. We demonstrate that the proposed coupled stochastic differential equations allow us to obtain a $1/f$ spectrum in a wide range of frequencies together with almost arbitrary steady-state PDF of the signal.

The paper is organized as follows: in section 2 we obtain a system of coupled SDEs generating signals with $1/f^{\beta}$ PSD by considering the scaling properties of the equations. Numerical methods of solution of such equations are discussed in section 3. SDEs obtained in section 2 do not have the most general form that is allowed by scaling properties required to get a $1/f^{\beta}$ spectrum. For completeness, in section 4 we consider a more general, but more complicated form of equations. Section 5 summarizes our findings.

2. Derivation of coupled stochastic differential equations using scaling properties

In this section we obtain a pair of coupled nonlinear SDEs by considering the scaling properties required to get a $1/f^{\beta}$ PSD. The method we use is similar to that in [41], however now we consider two stochastic variables and two equations. We assume that the first equation describes the fluctuations of the signal, with the fluctuating rate of change described by the second equation.

We can obtain a pair of coupled nonlinear SDEs generating signals exhibiting $1/f$ noise by using the following considerations. The Wiener–Khintchine theorem

$$C(t) = \int_0^{+\infty} S(f)\cos(2\pi f)\mathrm{d}f \tag{1}$$

relates the PSD $S(f)$ to the autocorrelation function $C(t)$. If the PSD has a power-law behavior $S(f) \sim f^{-\beta}$ in a wide range of frequencies $f_{\mathrm{min}} \ll f \ll f_{\mathrm{max}}$, then, when the influence of the limiting frequencies f_{min} and f_{max} is neglected, the PSD has a scaling property

$$S(af) \sim a^{-\beta}S(f) \tag{2}$$

for the frequencies in this range. In this paper we will consider signals with PSD having $1/f^{\beta}$ behavior only in some wide intermediate region of frequencies $f_{\min} \ll f \ll f_{\max}$. To avoid the divergence of the total power occurring for pure $1/f$ behavior at arbitrarily small frequencies, we assume that the PSD is bounded for small frequencies $f \ll f_{\min}$ outside of this region. Compatibility with experimental data can be ensured by choosing sufficiently small limiting frequency $f_{\min}$.

From the Wiener–Khintchine theorem (1) and equation (2) it follows that the autocorrelation function has the scaling property

$$C(at) \sim a^{\beta-1} C(t) \tag{3}$$

in the time range $1/f_{\max} \ll t \ll 1/f_{\min}$. Assuming that we have two stochastic variables x and y with the signal represented by the stochastic variable x, the autocorrelation function can be written as [42–44]

$$C(t) = \int \mathrm{d}x\mathrm{d}y \int \mathrm{d}x'\mathrm{d}y'\, xx' P_0(x, y) P(x', y', t|x, y, 0) - \left[\int \mathrm{d}x\mathrm{d}y\, x P_0(x, y)\right]^2 . \tag{4}$$

Here, $P_0(x, y)$ is the steady-state PDF and $P(x', y', t|x, y, 0)$ is the transition probability (the conditional probability that at time t the stochastic variables have values x' and y' with the condition that at time $t = 0$ they had had the values x and y). The transition probability can be obtained from the solution of the Fokker–Planck equation with the initial condition $P(x', y', 0|x, y, 0) = \delta(x - x')\delta(y - y')$. The last term in equation (4), being a constant, does not influence the PSD at frequencies $f > 0$. Therefore, we will neglect this term from now on.

One of the ways to obtain the required scaling property (3) is for the steady-state PDF to be a power-law function of the stochastic variable y,

$$P_0(x, y) \sim p(x) y^{-\lambda}, \tag{5}$$

and for the transition probability to have the scaling property

$$aP(x', ay, t|x, ay, 0) = P(x', y', a^{\mu}t|, x, y, 0). \tag{6}$$

Here, μ is the scaling exponent and λ is the power-law exponent of the steady-state PDF of the stochastic variable y. equation (6) means that the change of the magnitude of the stochastic variable $y \to ay$ is equivalent to the change of time scale $t \to a^{\mu}t$. Using equations (4)–(6) and performing a change of variables we get

$$C(at) = \int \mathrm{d}x\mathrm{d}y \int \mathrm{d}x'\mathrm{d}y' xx' P_0(x, y) P(x', y', at|x, y, 0) \tag{7}$$

$$\sim \int \mathrm{d}x\mathrm{d}y \int \mathrm{d}x'\mathrm{d}y' xx' p(x) y^{-\lambda} a^{\frac{1}{\mu}} P(x', a^{\frac{1}{\mu}} y', t|x, a^{\frac{1}{\mu}} y, 0) \tag{8}$$

$$\sim a^{\frac{\lambda-1}{\mu}} \int \mathrm{d}x\mathrm{d}u \int \mathrm{d}x'\mathrm{d}u' xx' p(x) u^{-\lambda} P(x', u', t|x, u, 0). \tag{9}$$

Therefore, the autocorrelation function has the required scaling property (3) with β given by

J. Stat. Mech. (2016) 043209

$$\beta = 1 + \frac{\lambda - 1}{\mu}\,. \tag{10}$$

We see that we obtain the pure $1/f$ noise when $\lambda = 1$.

In order to avoid the divergence of the steady-state PDF (5), the diffusion of stochastic variable y should be restricted at least from the side of small values. In general, equation (5) can hold only in some region $y_{\min} \ll y \ll y_{\max}$. When the diffusion of stochastic variable y is restricted, equation (6) cannot also be exact. However, if the influence of the limiting values $y_{\min}$ and $y_{\max}$ can be neglected for the time t in some region $t_{\min} \ll t \ll t_{\max}$, we can expect for the scaling (3) to be approximately valid in this time region.

To get the required scaling (6) of the transition probability, only powers of the stochastic variable y should enter into the pair of SDEs. Assuming that the coefficient in the noise term of the first equation is proportional to y^{η}, we will consider the following coupled Itô SDEs:

$$\mathrm{d}x_t = a(x_t) y_t^{2\eta} \mathrm{d}t + b(x_t) y^{\eta} \mathrm{d}W_t\,, \tag{11}$$

$$\mathrm{d}y_t = u(x_t) y_t^{2\eta+1} \mathrm{d}t + \sigma y_t^{\eta+1} \mathrm{d}W'_t\,. \tag{12}$$

Here, W_t and W'_t are standard Wiener processes. The parameter σ in equation (12) gives the intensity of the noise and the coefficient $u(x)$ needs to be determined. One can see that equations (11) and (12) indeed lead to the scaling of transition probability (6). Changing the variable y in (11), (12) to the scaled variable $y_{\mathrm{s}} = ay$ or introducing the scaled time $t_{\mathrm{s}} = a^{2\eta} t$ and using the property of the Wiener process $\mathrm{d}W_{t_{\mathrm{s}}} \stackrel{d}{=} a^{\eta} \mathrm{d}W_t$ we get the same resulting equations. Therefore, the change of the scale of the variable y and change of time scale are equivalent, as in equation (6), and the scaling exponent μ is equal to

$$\mu = 2\eta\,. \tag{13}$$

To ensure steady-state PDF (5) and for determination the unknown coefficient $u(x)$ in equation (12) we write the Fokker–Planck equation corresponding to the system of SDEs (11) and (12) [44]

$$\frac{\partial}{\partial t}P = -y^{2\eta}\frac{\partial}{\partial x}a(x)P - u(x)\frac{\partial}{\partial y}y^{2\eta+1}P + \frac{1}{2}y^{2\eta}\frac{\partial^2}{\partial x^2}b^2(x)P + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial y^2}y^{2\eta+2}P. \tag{14}$$

The steady-state PDF $P_0(x, y)$ is the solution of the equation

$$-y^{2\eta}\frac{\partial}{\partial x}a(x)P_0 - u(x)\frac{\partial}{\partial y}y^{2\eta+1}P_0 + \frac{1}{2}y^{2\eta}\frac{\partial^2}{\partial x^2}b^2(x)P_0 + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial y^2}y^{2\eta+2}P_0 = 0. \tag{15}$$

Equation (15) can be written in terms of the components of the probability current

$$J_x(x, y) = y^{2\eta}a(x)P_0 - \frac{1}{2}y^{2\eta}\frac{\partial}{\partial x}b^2(x)P_0, \tag{16}$$

J. Stat. Mech. (2016) 043209

$$J_y(x,y) = u(x)y^{2\eta+1}P_0 - \frac{1}{2}\sigma^2\frac{\partial}{\partial y}y^{2\eta+2}P_0 \tag{17}$$

as

$$\frac{\partial}{\partial x}J_x(x,y) + \frac{\partial}{\partial y}J_y(x,y) = 0\,. \tag{18}$$

Inserting equation (5) into (16) and (17) we get

$$J_x(x,y) = y^{2\eta-\lambda}\left[a(x)p(x) - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}b^2(x)p(x)\right], \tag{19}$$

$$J_y(x,y) = y^{2\eta+1-\lambda}p(x)\left[u(x) - \sigma^2\left(\eta+1-\frac{\lambda}{2}\right)\right]. \tag{20}$$

Assuming that the x-component of the probability current J_x should vanish at the reflective boundaries that are not parallel to x axis, then the expression in the square brackets in equation (19) should be zero for different values of y. Thus, the function $p(x)$ in (5) should be a solution to the differential equation

$$a(x)p(x) - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}b^2(x)p(x) = 0. \tag{21}$$

This equation means that the steady-state PDF of the stochastic variable x is determined only by the coefficients $a(x)$ and $b(x)$ of the SDE (11). Further, assuming that the y-component of the probability current J_y should vanish at the boundaries that are not parallel to the y axis, then the expression in the square brackets in equation (20) should be zero for different values of x. Therefore, $u(x) = \sigma^2(\eta + 1 - \lambda/2)$ and the required system of coupled SDEs is

$$\mathrm{d}x_t = a(x_t)y_t^{2\eta}\mathrm{d}t + b(x_t)y_t^{\eta}\mathrm{d}W_t\,, \tag{22}$$

$$\mathrm{d}y_t = \sigma^2\left(\eta+1-\frac{\lambda}{2}\right)y_t^{2\eta+1}\mathrm{d}t + \sigma y_t^{\eta+1}\mathrm{d}W'_t\,. \tag{23}$$

Note, that the second equation (23) has the form of nonlinear SDEs proposed in [35, 36]. Equations similar to (22), (23) have been considered in [45]. From equation (10) it follows that the power-law exponent in the PSD of the signal generated by the SDEs (22), (23) is related to the parameters η and λ as

$$\beta = 1 + \frac{\lambda-1}{2\eta}. \tag{24}$$

To get a stationary process and avoid the divergence of steady-state PDF, equation (23) should be considered together with boundaries restricting the diffusion of stochastic variable y or be modified. The simplest choice restricting the range of diffusion of the stochastic variable y is the reflective boundaries at $y = y_{\mathrm{min}}$ and $y = y_{\mathrm{max}}$. Another possibility is the modification of equation (23) to get rapidly decreasing steady-state PDF

J. Stat. Mech. (2016) 043209

when the stochastic variable y acquires values outside of the interval $[y_{\min}, y_{\max}]$. For example, the steady-state PDF

$$P_0(x,y) \sim p(x) y^{-\lambda} \exp\left\{ -\left(\frac{y_{\min}}{y}\right)^m - \left(\frac{y}{y_{\max}}\right)^m \right\} \tag{25}$$

with $m > 0$ has a power-law dependence on y when $y_{\min} \ll y \ll y_{\max}$ and exponential cut-offs when y is outside of the interval $[y_{\min}, y_{\max}]$. This exponentially restricted steady-state PDF is a result of the SDE

$$\mathrm{d}y_t = \sigma^2 \left(\eta + 1 - \frac{\lambda}{2} + \frac{m}{2}\left(\frac{y_{\min}^m}{y_t^m} - \frac{y_t^m}{y_{\max}^m}\right)\right) y_t^{2\eta+1}\mathrm{d}t + \sigma y_t^{\eta+1}\mathrm{d}W_t' \tag{26}$$

obtained from equation (23) by introducing additional terms into the drift.

2.1. Limiting frequencies

The restriction of the diffusion of the stochastic variable y to the interval $y_{\min} \ll y \ll y_{\max}$ makes the scaling (6) only approximate. As a result, the power-law part of the PSD is limited to a finite range of frequencies $f_{\min} \ll f \ll f_{\max}$. Let us estimate the limiting frequencies $f_{\min}$ and $f_{\max}$. The limiting values $y = y_{\min}$ and $y = y_{\max}$ should also participate in the scaling and equation (6) for the transition probability corresponding to SDEs (22) and (23) becomes

$$aP(x', ay, t | x, ay, 0; ay_{\min}, ay_{\max}) = P(x', y', a^\mu t |, x, y, 0; y_{\min}, y_{\max}). \tag{27}$$

Here, $y_{\min}$, $y_{\max}$ enter as parameters of the transition probability. Similarly, the steady-state PDF $P_0(x, y; y_{\min}, y_{\max})$ has the scaling property

$$aP_0(x, ay; ay_{\min}, ay_{\max}) = P_0(x, y; y_{\min}, y_{\max}). \tag{28}$$

Inserting equations (27) and (28) into (4) we get

$$C(t, ay_{\min}, ay_{\max}) = C(a^\mu t, y_{\min}, y_{\max}). \tag{29}$$

From this scaling of the autocorrelation function it follows that time t should enter only in combinations with the limiting values $y_{\min} t^{1/\mu}$ and $y_{\max} t^{1/\mu}$. We can expect that the influence of the limiting values can be neglected and the scaling (6) be approximately valid when $y_{\min} t^{1/\mu} \ll 1$ and $y_{\max} t^{1/\mu} \gg 1$. In other words, we expect that the scaling (6) holds when time t is in the interval $\sigma^{-2} y_{\max}^{-\mu} \ll t \ll \sigma^{-2} y_{\min}^{-\mu}$ when $\mu > 0$ and in the interval $\sigma^{-2} y_{\min}^{-\mu} \ll t \ll \sigma^{-2} y_{\max}^{-\mu}$ when $\mu < 0$. Using equation (1) the frequency range where the PSD has $1/f^\beta$ behavior can be estimated as

$$\sigma^2 y_{\min}^\mu \ll 2\pi f \ll \sigma^2 y_{\max}^\mu, \qquad \mu > 0 \tag{30}$$

$$\sigma^2 y_{\max}^\mu \ll 2\pi f \ll \sigma^2 y_{\min}^\mu, \qquad \mu < 0 \tag{31}$$

We see that the width of the frequency range where the PSD has $1/f^\beta$ behavior grows with an increase of the ratio $y_{\max}/y_{\min}$. For $\mu = 0$ (which corresponds to $\eta = 0$) the width of the frequency region (30) is zero and we do not have $1/f^\beta$ power spectral density.

3. Numerical approach

Since analytical solutions of stochastic differential equations can be obtained only in particular cases, there is a need for numerical solution. Using the Euler–Maruyama method with small time step Δt for numerical solutions of SDEs (22) and (23), we get the discretized equations

$$x_{k+1} = x_k + a(x_k) y_k^{2\eta} \Delta t + b(x_k) y_k^{\eta} \sqrt{\Delta t}\, \varepsilon_k, \tag{32}$$

$$y_{k+1} = y_k + \sigma^2 \left(\eta + 1 - \frac{\lambda}{2}\right) y_k^{2\eta+1} \Delta t + \sigma y_k^{\eta+1} \sqrt{\Delta t}\, \xi_k. \tag{33}$$

Here, ε_k and ξ_k are independent random variables with the standard normal distribution. However, for numerical solutions of nonlinear equations the solution schemes involving a fixed time step Δt can be inefficient. For example, in equations (22) and (23) with $\eta > 0$, large values of stochastic variable y lead to large coefficients and thus require a very small time step. The numerical solution scheme can be improved by using a variable time step that becomes small only when y becomes large. Such a method for the solution of a single nonlinear SDE has been proposed in [35, 46]. The variable time step is equivalent to the introduction of the internal time τ that is different from the real, physical, time t [46].

In order to make the solution more efficient we introduce an internal, operational, time τ by the equation

$$\mathrm{d}\tau_t = y_t^{2\eta} \mathrm{d}t\,. \tag{34}$$

We assume that the zero of the internal time τ coincides with the zero of the physical time t, thus the initial condition for the internal time is $\tau_{t=0} = 0$. Since $y_t > 0$, from equation (34) it follows that τ_t is a strictly increasing function of time t. Let us obtain the SDEs for the stochastic variables x and y in the internal time τ. To do this we proceed similarly as in [46] and consider the joint PDF $P_{x,y,\tau}(x, y, \tau; t)$ of the stochastic variables x, y and τ. The PDF $P(x, y;t)$ can be calculated using the equation

$$P_{x,y}(x, y, t) = \int P_{x,\tau}(x, y, \tau; t)\, \mathrm{d}\tau. \tag{35}$$

Equations (22), (23) and (34) lead to the Fokker–Planck equation for the PDF $P_{x,y,\tau}(x, y, \tau; t)$

$$\frac{\partial}{\partial t} P_{x,y,\tau} = -y^{2\eta} \frac{\partial}{\partial x} a(x) P_{x,y,\tau} - \sigma^2 \left(\eta + 1 - \frac{\lambda}{2}\right) \frac{\partial}{\partial y} y^{2\eta+1} P_{x,y,\tau} - y^{2\eta} \frac{\partial}{\partial \tau} P_{x,y,\tau} + \frac{1}{2} y^{2\eta} \frac{\partial^2}{\partial x^2} b(x)^2 P_{x,y,\tau} + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial y^2} y^{2\eta+2} P_{x,y,\tau}. \tag{36}$$

Since the zero of the internal time τ coincides with the zero of the physical time t, the initial condition for equation (36) is $P_{x,y,\tau}(x, y, \tau; 0) = P(x, y; 0)\delta(\tau)$. Matching of the zeros of τ and t also leads to the boundary condition $P_{x,y,\tau}(x, y, 0; t) = 0$ for $t > 0$, because τ and t are strictly increasing.

Instead of x, y and τ we can consider x, y and t as stochastic variables. The physical time t is related to the operational time τ via equation (34), therefore, the joint PDF $P_{x,y,t}(x, y, t; \tau)$ of the stochastic variables x, y and t is related to the PDF $P_{x,y,\tau}(x, y, \tau; t)$ according to the equation

$$P_{x,y,t}(x, y, t; \tau) = y^{2\eta} P_{x,y,\tau}(x, y, \tau; t). \tag{37}$$

Another way to get this relation is to recognize that the third term on the right-hand side of equation (36) contains the derivative $\frac{\partial}{\partial \tau}$ and thus should be equal to $-\frac{\partial}{\partial \tau} P_{x,y,t}$. Inserting (37) into equation (36) we get

$$\frac{\partial}{\partial \tau} P_{x,y,t} = -\frac{\partial}{\partial x} a(x) P_{x,y,t} - \sigma^2\left(\eta + 1 - \frac{\lambda}{2}\right)\frac{\partial}{\partial y} y P_{x,y,t} - \frac{\partial}{\partial t}\frac{1}{y^{2\eta}} P_{x,y,t} + \frac{1}{2}\frac{\partial^2}{\partial x^2} b(x)^2 P_{x,y,t} + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial y^2} y^2 P_{x,y,t}. \tag{38}$$

The initial condition for equation (38) is $P_{x,y,t}(x, t; 0) = P(x, y; 0)\delta(t)$. In addition, there is a boundary condition $P_{x,y,t}(x, y, 0; \tau) = 0$ for $\tau > 0$. The Fokker–Planck equation (38) can be obtained from the coupled SDEs:

$$\mathrm{d}x_\tau = a(x_\tau)\mathrm{d}\tau + b(x_\tau)\mathrm{d}W_\tau, \tag{39}$$

$$\mathrm{d}y_\tau = \sigma^2\left(\eta + 1 - \frac{\lambda}{2}\right) y_\tau \mathrm{d}\tau + \sigma y_\tau \mathrm{d}W'_\tau, \tag{40}$$

$$\mathrm{d}t_\tau = \frac{1}{y_\tau^{2\eta}}\mathrm{d}\tau. \tag{41}$$

Discretizing the internal time τ with the step $\Delta\tau$ and using the Euler–Maruyama approximation for SDEs (39) and (40), we get

$$x_{k+1} = x_k + a(x_k)\Delta\tau + b(x_k)\sqrt{\Delta\tau}\,\varepsilon_k, \tag{42}$$

$$y_{k+1} = y_k + \sigma^2\left(\eta + 1 - \frac{\lambda}{2}\right) y_k \Delta\tau + \sigma y_k \sqrt{\Delta\tau}\,\xi_k, \tag{43}$$

$$t_{k+1} = t_k + \frac{\Delta\tau}{y_k^{2\eta}}. \tag{44}$$

Equations (42)–(44) provide the numerical method for solving coupled SDEs (22) and (23). One can interpret equations (42)–(44) as an Euler–Maruyama scheme with a variable time step $\Delta t_k = \Delta\tau / y_k^{2\eta}$ that adapts to the coefficients in the SDEs. As a consequence of the introduction of the internal time the increments of the real, physical, time

t become random. To get the discretization of time with fixed steps the signal generated in such a way should be interpolated.

As an example, let us solve the equations

$$\mathrm{d}x_t = -\gamma y_t^{2\eta} x_t \mathrm{d}t + y_t^{\eta} \mathrm{d}W_t\,, \tag{45}$$

$$\mathrm{d}y_t = \sigma^2 \left(\eta + 1 - \frac{\lambda}{2}\right) y_t^{2\eta+1} \mathrm{d}t + \sigma y_t^{\eta+1} \mathrm{d}W'_t\,. \tag{46}$$

For the stochastic variable y we assume reflective boundaries at $y = y_{\mathrm{min}}$ and $y = y_{\mathrm{max}}$. In this case the coefficients $a(x)$ and $b(x)$ in equation (22) are $a(x) = -\gamma x$ and $b(x) = 1$, leading to the Gaussian steady-state PDF of x,

$$p(x) = \sqrt{\frac{\gamma}{\pi}}\, \mathrm{e}^{-\gamma x^2}. \tag{47}$$

The quantity $y^{2\eta}$ in equation (45) represents a fluctuating relaxation rate.

Comparison of the numerically obtained steady-state PDF and the PSD with analytical expressions for the system of SDEs (45) and (46) with $\eta = 1$ and $\lambda = 1$ is presented in figure 1. The typical signal x_t generated by equations (45) and (46) is shown in figure 1(a). As one can see, the signal exhibits a structure consisting of the periods of slow and fast fluctuations. The fast fluctuations correspond to the peaks or bursts of the stochastic variable y. Note, that due to a large difference between the slowest and fastest fluctuation rates the signal in the periods of fast fluctuations in figure 1(a) visually resembles white noise. However, the actual signal changes according to SDE (45), the periods of fast fluctuations are similar to the periods of slow fluctuations compressed in time. Analysis of nonlinear SDEs similar to (46), performed in [14], reveals that the sizes of the bursts are approximately proportional to the squared durations of the bursts. The distributions of burst and inter-burst durations have power-law parts, with the numerically estimated power-law exponent of the PDF of the inter-burst durations approximately equal to $-3/2$. Intermittent behavior, similar to the behavior shown in figure 1(a), can be connected with $1/f$ noise. For example, it is known that intermittent behavior in iterative maps at the edge of chaos can lead to $1/f$ noise [47]. In figures 1(b) and (c) we see a good agreement of the numerically calculated steady-state PDFs of the stochastic variables x and y with the analytical expressions. The PSD of the signal x_t is shown in figure 1(d). Numerical solutions of the equations confirm the presence of the frequency region for which the power spectral density has $1/f^{\beta}$ dependence with $\beta = 1$.

4. More general form of equations

Coupled nonlinear SDEs (22) and (23) exhibit the separation between the magnitude of the fluctuations of the signal x_t and the rate of fluctuations. The steady-state PDF of the signal is determined only by the coefficients $a(x)$ and $b(x)$ in equation (22), whereas equation (23) describes the fluctuating rate that does not depend on the signal. However, equations (22) and (23) are not the most general form of coupled SDEs that

J. Stat. Mech. (2016) 043209

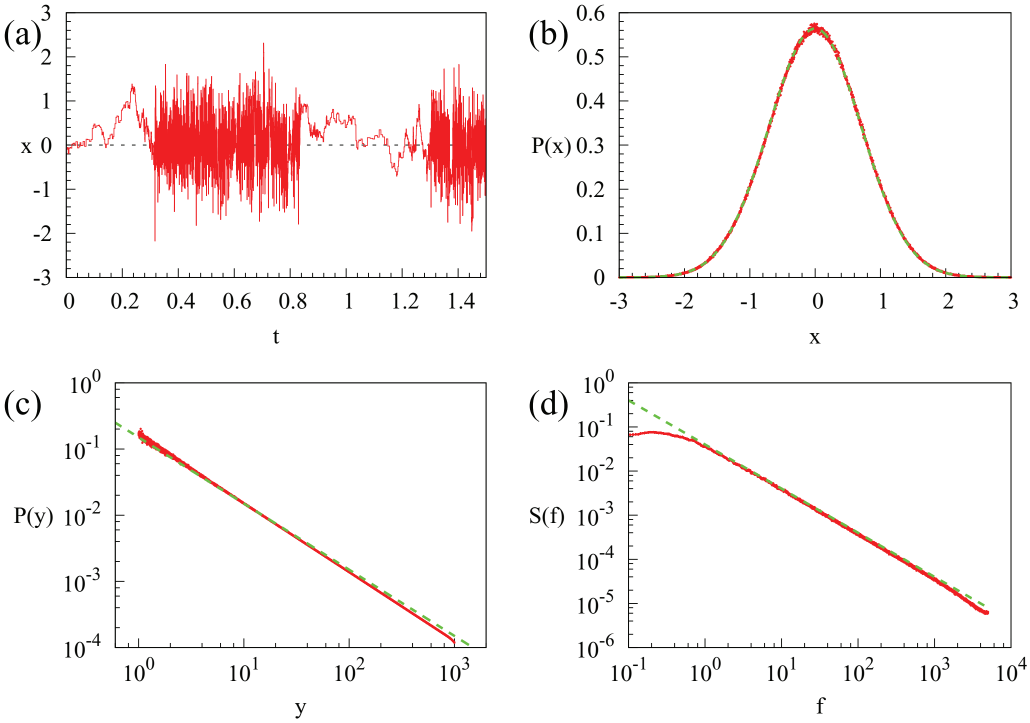

Figure 1. (a) Typical signal x generated by equations (45) and (46). Reflective boundaries at y_{min} and y_{max} have been used for equation (46). (b) The PDF of the signal intensity. The dashed (green) line shows the Gaussian curve. (c) The PDF of the stochastic variable y. The dashed (green) line shows the power-law with the exponent -1. (d) The PSD of the signal x. The dashed (green) line shows the slope f^{-1}. Parameters used are $\eta = 1$, $\lambda = 1$, $y_{\mathrm{min}} = 1$, $y_{\mathrm{max}} = 1000$, $\gamma = 1$ and $\sigma = 1$.

are allowed by scaling properties required to get a $1/f^{\beta}$ spectrum. For completeness, in this section we will consider a more general form of equations.

In general, scaling of time t in the transition probability can lead to scaling of both x and y, therefore instead of equation (6) in this section we will consider a more general scaling property of the transition probability:

$$a^{\rho+1}P(a^{\rho}x', ay, t|\, a^{\rho}x, ay, 0) = P(x', y', a^{\mu}t|, x, y, 0)\,. \tag{48}$$

We also assume a scaling property of the steady-state PDF similar to the scaling property (48) of the transition probability:

$$P_0(a^{\rho}x, ay) \sim a^{-\lambda}P_0(x, y)\,. \tag{49}$$

Here, μ, ρ and λ are the scaling exponents. From equation (49) it follows that the steady-state PDF should have the form

$$P_0(x, y) = p(xy^{-\rho})y^{-\lambda}, \tag{50}$$

where $p(\cdot)$ is an arbitrary function. Using equations (4), (48) and (49) and performing a change of variables we obtain

$$C(at) = \int \mathrm{d}x\mathrm{d}y \int \mathrm{d}x'\mathrm{d}y'xx'P_0(x, y)P(x', y', at|x, y, 0) \tag{51}$$

$$\sim \int \mathrm{d}x\mathrm{d}y \int \mathrm{d}x'\mathrm{d}y' x x' a^{\frac{\lambda}{\mu}} P_0(a^{\frac{\rho}{\mu}}x, a^{\frac{1}{\mu}}y) a^{\frac{\rho+1}{\mu}} P(a^{\frac{\rho}{\mu}}x', a^{\frac{1}{\mu}}y, t|a^{\frac{\rho}{\mu}}x, a^{\frac{1}{\mu}}y, 0) \tag{52}$$

$$\sim a^{\frac{\lambda-1-3\rho}{\mu}} \int \mathrm{d}u\mathrm{d}v \int \mathrm{d}u'\mathrm{d}v' u u' P_0(u, v) P(u', v, t|u, v, 0)\,. \tag{53}$$

Therefore, the autocorrelation function has the scaling property (3) required to get $1/f^{\beta}$ PSD, with the exponent β given by equation

$$\beta = 1 + \frac{\lambda - 1 - 3\rho}{\mu}. \tag{54}$$

In this case we obtain pure $1/f$ noise when $\lambda = 1 + 3\rho$.

To get the scaling property (48) of the transition probability, we will consider the following coupled Itô SDEs:

$$\mathrm{d}x_t = a(x_t y_t^{-\rho}) y_t^{2\eta+\rho} \mathrm{d}t + b(x_t y_t^{-\rho}) y_t^{\eta+\rho} \mathrm{d}W_t, \tag{55}$$

$$\mathrm{d}y_t = f(x_t y_t^{-\rho}) y_t^{2\eta+1} \mathrm{d}t + g(x_t y_t^{-\rho}) y_t^{\eta+1} \mathrm{d}W'_t. \tag{56}$$

Here, W_t and W'_t are standard Wiener processes. Note, that equations (55) and (56) do not have the most general form compatible with the scaling property (49), because in general both noises W_t and W'_t can affect both stochastic variables x and y. However, for simplicity we will not consider the most general case. One can see that equations (55) and (56) indeed lead to the scaling of transition probability (48). Changing the variables x and y in equations (55) and (56) to the scaled variables $x_{\mathrm{s}} = a^{\rho}x$ and $y_{\mathrm{s}} = ay$ or introducing the scaled time $t_{\mathrm{s}} = a^{2\eta t}$ and taking into account the property of the Wiener process $\mathrm{d}W_{t_{\mathrm{s}}} \overset{d}{=} a^{\eta}\mathrm{d}W_t$, we get the same resulting equations. Therefore, the change of the time scale is equivalent to the corresponding change of scale of the variables x and y, according to equation (48) with the scaling exponent $\mu = 2\eta$.

We will determine the connection between the coefficients $f(\cdot)$ and $g(\cdot)$ by requiring the steady-state PDF of the form (50). The Fokker–Planck equation corresponding to the SDEs (55) and (56) is

$$\begin{aligned} \frac{\partial}{\partial t}P = & -y^{2\eta+\rho}\frac{\partial}{\partial x}a(xy^{-\rho})P - \frac{\partial}{\partial y}f(xy^{-\rho})y^{2\eta+1}P \\ & + \frac{1}{2}y^{2\eta+2\rho}\frac{\partial^2}{\partial x^2}b^2(xy^{-\rho})P + \frac{1}{2}\frac{\partial^2}{\partial y^2}g^2(xy^{-\rho})y^{2\eta+2}P, \end{aligned} \tag{57}$$

therefore, the steady-state PDF $P_0(x, y)$ is the solution of the equation

$$-y^{2\eta+\rho}\frac{\partial}{\partial x}a(xy^{-\rho})P - \frac{\partial}{\partial y}f(xy^{-\rho})y^{2\eta+1}P + \frac{1}{2}y^{2\eta+2\rho}\frac{\partial^2}{\partial x^2}b^2(xy^{-\rho})P + \frac{1}{2}\frac{\partial^2}{\partial y^2}g^2(xy^{-\rho})y^{2\eta+2}P = 0. \tag{58}$$

Equation (58) can be written in terms of the components of the probability current

$$J_x(x,y)=y^{2\eta+\rho}a(xy^{-\rho})P_0-\frac{1}{2}y^{2\eta+2\rho}\frac{\partial}{\partial x}b^2(xy^{-\rho})P_0, \quad (59)$$

$$J_y(x,y)=f(xy^{-\rho})y^{2\eta+1}P_0-\frac{1}{2}\frac{\partial}{\partial y}g^2(xy^{-\rho})y^{2\eta+2}P_0. \quad (60)$$

Inserting steady-state PDF (50) into equations (59) and (60) we get

$$J_x(x,y)=y^{2\eta+\rho-\lambda}\left[a(xy^{-\rho})p(xy^{-\rho})-\frac{1}{2}y^{\rho}\frac{\partial}{\partial x}b^2(xy^{-\rho})p(xy^{-\rho})\right], \quad (61)$$

$$J_y(x,y)=y^{2\eta+1-\lambda}g^2(xy^{-\rho})p(xy^{-\rho})\times\left[\frac{f(xy^{-\rho})}{g^2(xy^{-\rho})}-\eta-1+\frac{\lambda}{2}+\rho xy^{-\rho}\left(\frac{g'(xy^{-\rho})}{g(xy^{-\rho})}+\frac{1}{2}\frac{p'(xy^{-\rho})}{p(xy^{-\rho})}\right)\right]. \quad (62)$$

Assuming that the x-component of the probability current J_x should vanish at the boundaries that are not parallel to the x axis, then the expression in the square brackets in equation (61) should be zero for different values of y. Therefore, the function $p(\cdot)$ should be a solution to the differential equation

$$a(z)p(z)-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z}b^2(z)p(z)=0. \quad (63)$$

This equation means that the function $p(\cdot)$ in equation (50) is determined only by the coefficients of equation (55). Similarly, assuming that the y-component of the probability current J_y should vanish at the boundaries that are not parallel to the y axis then the expression in the square brackets in equation (62) should be zero for different values of y. Therefore, the coefficient $f(\cdot)$ is related to the coefficients $a(\cdot)$, $b(\cdot)$ and $g(\cdot)$ via the equation

$$f(z)=\left[\eta+1-\frac{\lambda}{2}-\rho z\left(\frac{g'(z)}{g(z)}+\frac{1}{2}\frac{p'(z)}{p(z)}\right)\right]g^2(z). \quad (64)$$

Let us consider some particular choices of the coefficients $f(\cdot)$ and $g(\cdot)$ in equation (56). According to equation (64), constant coefficient $g(z)=\sigma=\text{const.}$ leads to

$$f(z)=\sigma^2\left[\eta+1-\frac{\lambda}{2}-\rho z\left(\frac{a(z)}{b^2(z)}-\frac{b'(z)}{b(z)}\right)\right]. \quad (65)$$

Here, we used equation (63) for the function $p(z)$. When

$$\frac{g'(z)}{g(z)}+\frac{1}{2}\frac{p'(z)}{p(z)}=0, \quad (66)$$

From equation (64) it follows that the stochastic variable x enters into the coefficients $f(\cdot)$ and $g(\cdot)$ only as an argument of the function $p(\cdot)$. The solution of equation (66) is

$g(z)=\sigma p(z)^{-1/2}$. Consequently, $f(z)=\sigma^2(\eta+1-\lambda/2)p(z)^{-1}$ and equations (61) and (62) take the form

$$\mathrm{d}x_t=a(x_t y_t^{-\rho})y_t^{2\eta+\rho}\mathrm{d}t+b(x_t y_t^{-\rho})y_t^{\eta+\rho}\mathrm{d}W_t\,, \tag{67}$$

$$\mathrm{d}y_t=\sigma^2\left(\eta+1-\frac{\lambda}{2}\right)\frac{y_t^{2\eta+1}}{p(x_t y_t^{-\rho})}\mathrm{d}t+\frac{\sigma y_t^{\eta+1}}{\sqrt{p(x_t y_t^{-\rho})}}\mathrm{d}W'_t. \tag{68}$$

As an example, let us take the SDE (55) describing the fluctuations of the signal x:

$$\mathrm{d}x_t=-y_t x_t\mathrm{d}t+y_t^{\nu}\mathrm{d}W_t\,. \tag{69}$$

The stochastic variable y in equation (69) represents a fluctuating relaxation rate. The value of $\nu=\frac{1}{2}$ corresponds to the fluctuation-dissipation theorem. However, there are some cases where the fluctuation-dissipation theorem cannot be applied and other values of ν are possible. The violation of the fluctuation-dissipation theorem has been found in the finite dimensional spin glasses [48] and in the systems out of equilibrium [49]. The theoretical study of motion of colloidal particles being confined in a harmonic well and dragged by a shear flow also shows violation of the fluctuation-dissipation theorem [50]. Comparing equation (69) with equation (55) we have $a(z)=-z$, $b(z)$, $\eta=\frac{1}{2}$, $\rho=\nu-\frac{1}{2}$. Using equations (56) and (65) we obtain the second equation:

$$\mathrm{d}y_t=\sigma^2\left(\frac{3}{2}-\frac{\lambda}{2}+\left(\nu-\frac{1}{2}\right)y_t^{1-2\nu}x_t^2\right)y_t^2\mathrm{d}t+\sigma y_t^{\frac{3}{2}}\mathrm{d}W'_t. \tag{70}$$

According to (54), equations (69) and (70) generate the signal x_t with power-law behavior $1/f^{\beta}$ of the PSD in a wide range of frequencies, with the exponent $\beta=\lambda+3\left(\frac{1}{2}-\nu\right)$.

As an another example let us consider the SDE (55) with the coefficients $a(z)=0$ and $b(z)=\mathrm{const}$:

$$\mathrm{d}x_t=by_t^{\eta+\rho}\mathrm{d}\,W_t\,. \tag{71}$$

To get a stationary solution of the corresponding Fokker–Planck equation, equation (71) should be taken together with boundaries limiting the region of diffusion of stochastic variable x. For such coefficients $a(z)$ and $b(z)$ the solution of equation (63) is $p(z)=\mathrm{const}$.. Equations (56) and (65) yield the second SDE

$$\mathrm{d}y_t=\sigma^2\left(\eta+1-\frac{\lambda}{2}\right)y_t^{2\eta+1}\mathrm{d}t+\sigma y_t^{\eta+1}\mathrm{d}W'_t. \tag{72}$$

We see that in this case the second equation does not depend on x.

5. Discussion and conclusions

Coupled Langevin equations have been used to describe many physical phenomena. For example, hot-carrier transport in semiconductors has been modeled by linearly coupled Langevin equations [51]; nonlinear coupled Langevin equations have been used

J. Stat. Mech. (2016) 043209

to study pressure time series [52]. One nonlinear SDE with fluctuating parameters can be interpreted as a pair of coupled SDEs [53]. Equations with a time varying parameter being a Gaussian colored noise (Ornstein–Uhlenbeck process) have been used to model wind farm power production output dependence on wind velocity [54] and atmospheric turbulence in radio signal detection [55]. In this paper we study nonlinear SDEs where the fluctuating parameter enters both diffusion and drift coefficients as a power-law function.

Coupled SDEs are also used in finance and econophysics for stochastic volatility models [56]; some of those models correspond to equations presented in section 4. For example, SDE (71) and SDE (72) with an additional drift term causing exponential restriction of the steady-state PDF, when the parameters η and ρ take values $\eta = -\frac{1}{2}$, $\rho = 1$ have the form of the Heston model: [57]

$$\mathrm{d}x_t = \sqrt{y_t}\,\mathrm{d}W_t\,, \tag{73}$$

$$\mathrm{d}y_t = \frac{1}{2}\sigma^2\left(1 - \lambda - \frac{y_t}{y_{\max}}\right)\mathrm{d}t + \sigma\sqrt{y_t}\mathrm{d}W'_t\,. \tag{74}$$

In this model the stochastic variable x represents the logarithm of the price and the stochastic variable y is the volatility.

To illustrate the situation that can be described by the proposed SDEs (22) and (23), let us consider the case with $\eta = -\frac{1}{2}$. Equations (22) and (23) then become

$$\mathrm{d}x_t = a(x_t)\frac{1}{y_t}\mathrm{d}t + b(x_t)\frac{1}{\sqrt{y_t}}\mathrm{d}W_t\,, \tag{75}$$

$$\mathrm{d}y_t = \frac{1}{2}\sigma^2(1-\lambda)\mathrm{d}t + \sigma\sqrt{y_t}\mathrm{d}W'_t\,. \tag{76}$$

The quantity y^{-1} in equation (75) has the meaning of the rate of change, whereas y has the meaning of the time interval. According to equation (54), the PSD of the signal x_t has power-law behavior for a wide range of frequencies with the power-law exponent

$$\beta = 2 - \lambda\,. \tag{77}$$

We get $1/f$ noise when $\lambda = 1$. Assuming that the coefficients $a(x)$ and $b(x)$ are sufficiently small, we can take $\Delta\tau = 1$ in the numerical solution scheme (42)–(44), leading to the discrete equations

$$x_{k+1} = x_k + a(x_k) + b(x_k)\varepsilon_k\,, \tag{78}$$

$$y_{k+1} = y_k\left(1 + \frac{1}{2}\sigma^2(1-\lambda) + \sigma\xi_k\right), \tag{79}$$

$$t_{k+1} = t_k + y_k. \tag{80}$$

In particular, when $\lambda = 1$ and the signal x have a $1/f$ spectrum, equation (79) becomes $y_{k+1} = y_k(1 + \sigma\xi_k)$. We can interpret equations (78)–(80) as follows: equations (79) and (80) describe a process consisting of discrete events occurring at time moments t_k. The inter-event duration is random and equal to the stochastic variable y_k. This inter-event duration slowly changes with time in such a way that the duration of the next time interval is equal to the duration of the previous interval multiplied by some random factor close to 1. The signal x_k changes only during the occurrence of the events at time moments t_k and this change is described by equation (78).

Equation (76) results in the steady-state PDF $P_0(y_t)$ of the stochastic variable y_t having a power-law form with the exponent $-\lambda$. The PDF $P_k(y_k)$ of a sequence of y_k values generated according to equation (79) differs from $P_0(y_t)$. When y_k changes slowly with the index k, the PDF $P_k(y_k)$ should satisfy the equation $P_0(y_k) \approx \frac{y_k}{\langle y_k \rangle} P_k(y_k)$, because going back from discrete equations to the continuous time one should assume that each value y_k lasts for the duration also equal y_k. Consequently, the PDF $P_k(y_k)$ is also a power-law with the exponent $-\lambda'$, $\lambda' = \lambda + 1$. Thus, if λ is close to 1 then λ' is close to 2.

There are many processes in nature with the power-law inter-event time distribution. For example, many human-related activities show power-law decaying inter-event time distribution with exponents usually varying between 1 and 2 [58–61]. Power-law distribution of inter-event times has been observed in neuron-firing sequences [62] and in the timings of earthquakes [63, 64]. In addition, power-law decaying inter-event time distribution is often accompanied by the power-law decaying autocorrelation function [65].

Let us further assume that the events are due to jumps over the potential barrier of the height v. In many physical systems the escape rate exponentially depends on the barrier height; therefore we take $y = \mathrm{e}^v$. Changing the variables in equations (75) and (76) we get the SDEs

$$\mathrm{d}x_t = a(x_t)\mathrm{e}^{-v_t}\mathrm{d}t + b(x_t)\mathrm{e}^{-v_t/2}\mathrm{d}W_t\,, \tag{81}$$

$$\mathrm{d}v_t = -\frac{1}{2}\sigma^2\lambda\mathrm{e}^{-v_t}\mathrm{d}t + \sigma\mathrm{e}^{-v_t/2}\mathrm{d}W'_t\,. \tag{82}$$

Similar to equations (78)–(80), a numerical solution scheme with the variable time step $\Delta t_k = \mathrm{e}^{v_k}$ yields discrete equations

$$x_{k+1} = x_k + a(x_k) + b(x_k)\varepsilon_k\,, \tag{83}$$

$$v_{k+1} = v_k - \frac{1}{2}\sigma^2\lambda + \sigma\xi_k\,, \tag{84}$$

$$t_{k+1} = t_k + \mathrm{e}^{v_k}\,. \tag{85}$$

From equation (84) we see that the potential v performs a simple random walk with a constant drift. When the potential has the value v_k, the time interval that one needs to wait till the next event is e^{v_k}. Both signal x and the potential v change during the jump at time moment t_k. One can also consider the case where the time interval between events is random, with the average equal to e^{v_k}. We can expect that the randomness of the time interval should not change the PSD of the signal x_t at low frequencies.

In conclusion, we have proposed a pair of coupled nonlinear SDEs (22) and (23) that generate the signal x_t having the power-law PSD $S(f) \sim f^{-\beta}$ in an arbitrarily wide range of frequencies. The exponent β is given by equation (24). In contrast to a single nonlinear SDE generating $f^{-\beta}$ noise, the signal x_t generated by the proposed pair of SDEs can have almost arbitrary steady-state PDF. The steady-state PDF of the signal x_t is determined only by the coefficients $a(x)$ and $b(x)$ of the first SDE (22). One can interpret the first equation (22) as describing the fluctuations of the signal, with the fluctuating rate of change, described by the second equation (23). Thus, the proposed SDEs exhibit a separation between the magnitude of the fluctuations of the signal x_t and the rate of fluctuations. We expect that the proposed equations will be useful for the description of $1/f$ noise in various physical and social systems. In addition, the equations can be used for numerical generation of $1/f$ noise with the desired steady-state PDF of the signal.

References

[1] Johnson J B 1925 *Phys. Rev.* **26** 71
[2] Schottky W 1926 *Phys. Rev.* **28** 74
[3] Bernamont J 1934 *C. R. Acad. Sci., Paris* 198 1755, 2144
[4] Bernamont J 1937 *Ann. Physique* **7** 71
[5] Bernamont J 1937 *Proc. Phys. Soc. Lond.* **49** 138
[6] McWhorter A L 1957 *Semiconductor Surface Physics* ed R H Kingston (Philadelphia: University of Pennsylvania Press) pp 207–28
[7] Hooge F N, Kleinpenning T G M and Vadamme L K J 1981 *Rep. Prog. Phys.* **44** 479
[8] Weissman M B 1988 *Rev. Mod. Phys.* **60** 537
[9] Mandelbrot B B 1999 *Multifractals and 1/f Noise: Wild Self-Affinity in Physics* (New York: Springer)
[10] Ward L M and Greenwood P E 2007 *Scholarpedia* **2** 1537
[11] Wong H 2003 *Microelectron. Reliab.* **43** 585
[12] Kogan S 2008 *Electronic Noise and Fluctuations in Solids* (Cambridge: Cambridge University Press)
[13] Balandin A A 2013 *Nat. Nanotechnol.* **8** 549
[14] Kaulakys B and Alaburda M 2009 *J. Stat. Mech.* **2009** P02051
[15] Ralls K S, Skocpol W J, Jackel L D, Howard R E, Fetter L A, Epworth R W and Tennant D M 1984 *Phys. Rev. Lett.* **52** 228
[16] Rogers C T and Buhrman R A 1984 *Phys. Rev. Lett.* **53** 1272
[17] Kaulakys B, Gontis V and Alaburda M 2005 *Phys. Rev.* E **71** 051105
[18] Watanabe S 2005 *J. Korean Phys. Soc.* **46** 646
[19] Dutta P and Horn P M 1981 *Rev. Mod. Phys.* **53** 497
[20] Van Vliet C M 1991 *Solid-State Electron.* **34** 1
[21] Erland S, Greenwood P E and Ward L M 2011 *Europhys. Lett.* **95** 60006
[22] Bak P, Tang C and Wiesenfeld K 1987 *Phys. Rev. Lett.* **59** 381
[23] Bak P 1996 *How Nature Works: the Science of Self-Organized Criticality* (New York: Copernicus)
[24] Jensen H J, Christensen K and Fogedby H C 1989 *Phys. Rev.* B **40** 7425
[25] Kertesz J and Kiss L B 1990 *J. Phys. A: Math. Gen.* **23** L443
[26] Ali A A 1995 *Phys. Rev.* E **52** R4595
[27] Maslov S, Tang C and Zhang Y C 1999 *Phys. Rev. Lett.* **83** 2449
[28] Lowen S B and Teich M C 2005 *Fractal-Based Point Processes* (New York: Wiley)
[29] Chamberlin R V and Nasir D M 2014 *Phys. Rev.* E **90** 012142
[30] Orlyanchik V, Weissman M B, Torija M A, Sharma M and Leighton C 2008 *Phys. Rev.* B **78** 094430
[31] Melkonyan S V 2010 *Physica* B **405** 379
[32] Kaulakys B and Meškauskas T 1998 *Phys. Rev.* E **58** 7013
[33] Kaulakys B 1999 *Phys. Lett.* A **257** 37

[34] Kaulakys B 2000 *Microelectron. Reliab.* **40** 1787
[35] Kaulakys B and Ruseckas J 2004 *Phys. Rev.* E **70** 020101
[36] Kaulakys B, Ruseckas J, Gontis V and Alaburda M 2006 *Physica* A **365** 217
[37] Gontis V, Ruseckas J and Kononovicius A 2010 *Physica* A **389** 100–6
[38] Mathiesen J, Angheluta L, Ahlgren P T H and Jensen M H 2013 *Proc. Natl Acad. Sci.* **110** 17259
[39] Ton R and Daffertshofer A 2016 *NeuroImage* at press
[40] Li M and Zhao W 2012 *Math. Probl. Eng.* **2012** 673648
[41] Ruseckas J and Kaulakys B 2014 *J. Stat. Mech.* P06005
[42] Ruseckas J and Kaulakys B 2010 *Phys. Rev.* E **81** 031105
[43] Risken H and Frank T 1996 *The Fokker–Planck Equation: Methods of Solution and Applications* (Berlin: Springer)
[44] Gardiner C W 2004 *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences* (Berlin: Springer)
[45] Kaulakys B, Kazakevicius R and Ruseckas J 2013 Modeling Gaussian and non-Gaussian 1/*f* noise by the linear stochastic differential equations *22nd Int. Conf. on Noise and Fluctuations* pp 1–4
[46] Ruseckas J, Kazakevicius R and Kaulakys B 2015 1/*f* noise from point process and time-subordinated Langevin equations *J. Stat. Mech.* at press (arXiv:1512.03910[cond-mat.stat-mech])
[47] Schuster H G 1988 *Deterministic Chaos* (Weinheim: VCH)
[48] Marinari E, Parisi G, Ricci-Tersenghi F and Ruiz-Lorenzo J J 1998 *J. Phys. A: Math. Gen.* **31** 2611
[49] Lobaskin D and Kehrein S 2006 *J. Stat. Phys.* **123** 301
[50] Mauri R and Leporini D 2006 *Europhys. Lett.* **76** 1022
[51] Kuhn T, Reggiani L and Varani L 1992 *Phys. Rev.* B **45** 1903
[52] Lind P G, Mora A, Gallas J A C and Haase M 2005 *Phys. Rev.* E **72** 056706
[53] Jizba P and Kleinert H 2008 *Phys. Rev.* E **78** 031122
[54] Milan P, Wächter M and Peinke J 2014 *J. Renew. Sustain. Energy* **6** 03311
[55] Kloeden P E and Platen E 1992 *Numerical Solution of Stochastic Differential Equations* (Berlin: Springer)
[56] Slanina F 2014 *Essentials of Econophysics Modelling* (New York: Oxford University Press)
[57] Heston S L 1993 *Rev. Financ. Stud.* **6** 327
[58] Eckmann J P, Moses E and Sergi D 2004 *Proc. Natl Acad. Sci. USA* **101** 14333
[59] Oliveira J G and Barabási A L 2005 *Nature* **437** 1251
[60] Dezsö Z, Almaas E, Lukács A, Rácz B, Szakadát I and Barabási A L 2006 *Phys. Rev.* E **73** 066132
[61] Vázquez A, Oliveira J G, Dezsö Z, Goh K I, Kondor I and Barabási A L 2006 *Phys. Rev.* E **73** 036127
[62] Kemuriyama T, Ohta H, Sato Y, Maruyama S, Tandai-Hiruma M, Kato K and Nishida Y 2010 *BioSystems* **101** 144–7
[63] Corral Á 2004 *Phys. Rev. Lett.* 92 108501
[64] Godano C 2015 *Geophys. J. Int.* **202** 219–23
[65] Karsai M, Kaski K, Barabási A L and Kertész J 2012 *Sci. Rep.* **2** 397

J. Stat. Mech. (2016) 043209