

$1/f$ noise from point process and time-subordinated Langevin equations

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Abstract. Internal mechanism leading to the emergence of the widely occurring $1/f$ noise still remains an open issue. In this paper we investigate the distinction between the internal time of the system and the physical time as a source of $1/f$ noise. After demonstrating the appearance of $1/f$ noise in the earlier proposed point process model, we generalise it starting from a stochastic differential equation which describes a Brownian-like motion in the internal (operational) time. We consider this equation together with an additional equation relating the internal time to the external (physical) time. We show that the relation between the internal time and the physical time that depends on the intensity of the signal can lead to $1/f$ noise in a wide interval of frequencies. The present model can be useful for the explanation of the appearance of $1/f$ noise in different systems.

Keywords: driven diffusive systems (theory), stochastic processes (theory), stochastic processes, current fluctuations

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1. Introduction

The $1/f$ noise is a random process described by the power spectral density (PSD), $S(f)$, roughly proportional to the reciprocal frequency, $1/f$, i.e. $S(f) \propto 1/f^\beta$, with β close to 1. It was observed first as an excess low-frequency noise in vacuum tubes [1, 2], later in condensed matter [3–7] and other systems [8–10]. The general nature of $1/f$ noise (named also ‘flicker noise’ and ‘ $1/f$ fluctuations’) has up to now been the subject of several discussions and investigations, see [10–13] for review.

Many models have been proposed to explain the origin of $1/f$ noise. A short discussion about the models and theories of $1/f$ noise is available in the introduction of paper [14]. The widely used model of $1/f$ noise interprets the spectrum as a superposition of Lorentzians with a wide range distribution of relaxation times [5, 6, 15, 16]. Another possibility to model signals and processes featuring $1/f^\beta$ noise is a representation of the signals as consisting of the renewal pulses or events with the power-law distribution of the inter-event time [17].

A class of models of $1/f$ noise relevant for driven nonequilibrium systems involves self-organised criticality (SOC) [18–21]. SOC refers to the tendency of nonequilibrium systems driven by slow constant energy input to organise themselves into a correlated state where all scales are relevant [19]. In [18] a simple driven automaton model of sand piles that reaches a state characterised by power-law time and space correlations was introduced. However, the mechanism of self-organised criticality does not necessarily result in $1/f^\beta$ fluctuations with β close to 1 [22, 23]. The $1/f$ noise in the fluctuations of a mass was first seen in a sand pile model with threshold dissipation, proposed in [24]. In addition, the exponent β is exactly 1 in the spectrum of fluctuations of mass in a one-dimensional directed model of sand piles [25].

In most cases the $1/f$ noise is a Gaussian process [12, 26], although sometimes $1/f$ fluctuations are non-Gaussian [27, 28]. Processes with the power-law distributions of the signal characteristics can be modelled by presuming that the time between the adjacent pulses experience slow (the change from one inter-pulse duration to the next much smaller than the duration itself) Brownian-like motion [29–31]. Moreover, the nonlinear stochastic differential equations (SDEs) generating $1/f^\beta$ noise have been obtained and analysed [14, 32, 33] starting from this point process model. SDE generating $1/f$ noise should necessarily be nonlinear, because systems of linear SDEs do not generate signals with a $1/f$ spectrum. Such nonlinear SDEs have been applied to describe signals in socio-economical systems [34, 35].

In the signal consisting of a sequence of pulses the pulse number is a progressively increasing quantity and it can be understood as an internal time of the process. The purpose of this paper is to investigate the distinction between the internal time of the system and the physical time in connection with $1/f$ noise. We intend to generalise the mechanism leading to $1/f$ noise in the point process model, proposed in [29–31]. Instead of a sequence of pulses we start from an SDE describing a Brownian-like motion. We compose a new equation by interpreting the time in the SDE as an internal parameter and adding an additional equation relating the internal time to the physical time. We demonstrate that the relation between the internal time and the external time, depending on the intensity of the signal, can lead to $1/f$ noise in a wide interval of frequencies.

A process $x(\tau(t))$ obtained by randomising the time clock of a random process $x(t)$ using a new clock $\tau(t)$, where $\tau(t)$ is a random process with non-negative increments, is called the subordinated process [36]. The process $\tau(t)$ is referred to as a directing process, randomised time or operational time. In physics the time-subordinated equations have been applied to describe anomalous diffusion. Fogedby [37] introduced a class of coupled Langevin equations consisting of a Langevin process $x(s)$ in a coordinate s and a Lévy process representing a stochastic relation $t(s)$. This class of coupled Langevin equations was further investigated in [38], where N -time joint probability distributions were analysed. Properties of the inverse α -stable subordinator were considered in [39, 40]. It was shown [41, 42] that the description of anomalous diffusion by a Markovian dynamics governed by an ordinary Langevin equation but proceeding in an auxiliary, operational time instead of the physical time is equivalent to a fractional Fokker–Planck equation. A numerical simulation of subordinated equations was explored in [42, 43].

In contrast to the description of the anomalous diffusion, in this paper we consider the situation when small increments of the physical time are proportional to the increments of the operational time, with the the coefficient of proportionality that depends on the stochastic variable x representing the signal intensity. Thus, in our case the randomness of the operational time comes from the randomness of x .

The paper is organised as follows: in section 2 we briefly present the point process model of $1/f$ noise and obtain the PSD of the signal by a new method. In section 3 we generalise the mechanism leading to $1/f$ noise presented in section 2. We introduce the difference between the physical and the internal time and consider time-subordinated Langevin equations. In section 4 we examine several stochastic processes and, introducing the internal and external times, we check whether $1/f$ noise can be obtained. In section 5 we discuss a way of solving highly non-linear SDEs by introducing suitably

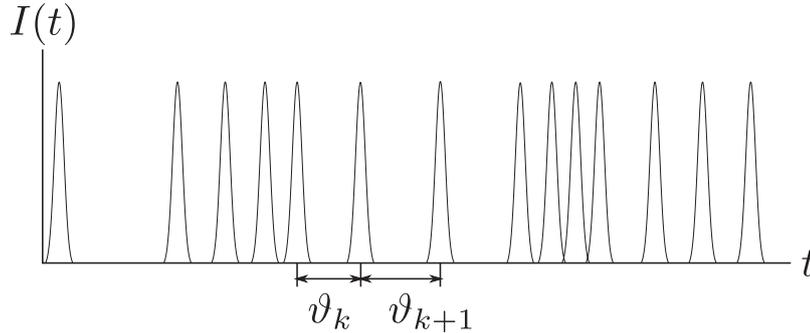


Figure 1. Sequence of pulses with random inter-pulse durations ϑ_k .

chosen internal time and the variable step of integration. Section 6 summarises our findings.

2. 1/f noise in a signal consisting of pulses

One of the models of 1/f noise was presented in [29–31]. In this model a signal consisted of pulses with the time between adjacent pulses undergoing a Brownian-like motion. It was shown that this Brownian-like motion of the inter-pulse durations can yield 1/f noise. In this section we briefly present this model and obtain the PSD of the signal using a different method than the method used in [29–31]. The new method allows us to better estimate the frequency range where the PSD has 1/f behaviour.

Let us consider a signal consisting of a pulse sequence having correlated inter-pulse durations. We assume that: (i) the pulse sequences are stationary and ergodic; (ii) all the pulses are described by the same shape function $A(t)$. The general form of the signal can be written as

$$I(t) = \sum_k A(t - t_k), \tag{1}$$

where the functions $A(t)$ determine the shape of the individual pulse and time moments t_k determine when the pulse occurs. The inter-pulse duration is $\vartheta_k = t_{k+1} - t_k$. This pulse sequence is schematically shown in figure 1.

The PSD of this signal is given by the equation

$$S(f) = \lim_{T \rightarrow \infty} \left\langle \frac{2}{T} \left| \int_{t_i}^{t_f} I(t) e^{-i2\pi ft} dt \right|^2 \right\rangle, \tag{2}$$

where $T = t_f - t_i$ is the observation time and the brackets $\langle \cdot \rangle$ denote the averaging over realisations of the pulse sequence. Note that in equation (2) we consider one-sided PSD, thus we have multiplier 2 in it. Introducing the Fourier transform $F(\omega)$ of the pulse shape function $A(t)$, we can write equation (1) as

$$S(f) = |F(\omega)|^2 \lim_{T \rightarrow \infty} \left\langle \frac{2}{T} \left| \sum_k e^{-i\omega t_k} \right|^2 \right\rangle. \tag{3}$$

Here $\omega = 2\pi f$. If the pulses are narrow and we are considering low frequencies then the Fourier transform $F(\omega)$ of the pulse shape is almost constant. In this case we can replace the actual pulses with δ -functions and drop $F(\omega)$ in the equations.

The PSD can be decomposed into two parts,

$$S(f) = \lim_{T \rightarrow \infty} \left\langle \frac{2}{T} \sum_{k,k'} e^{i\omega(t_{k'} - t_k)} \right\rangle \quad (4)$$

$$= \lim_{T \rightarrow \infty} \left\langle \frac{2}{T} \sum_k 1 \right\rangle + \lim_{T \rightarrow \infty} \left\langle \frac{2}{T} \left(\sum_{k' > k} e^{i\omega(t_{k'} - t_k)} + \sum_{k > k'} e^{i\omega(t_{k'} - t_k)} \right) \right\rangle \quad (5)$$

$$\equiv S_1(f) + S_2(f). \quad (6)$$

The first term can be written as

$$S_1(f) = 2\nu, \quad (7)$$

where ν is the mean number of pulses per unit time. By changing k into k' in the second part of the PSD one sees that it can be expressed as

$$S_2(f) = 4\text{Re} \lim_{T \rightarrow \infty} \left\langle \frac{1}{T} \sum_{k' > k} e^{i\omega(t_{k'} - t_k)} \right\rangle, \quad (8)$$

where the time difference $t_{k'} - t_k$ is

$$t_{k'} - t_k = \sum_{q=k}^{k'-1} \vartheta_q. \quad (9)$$

Thus, equation (5) becomes

$$S(f) = 2\nu + 4\nu \text{Re} \sum_{q=1}^{\infty} \left\langle e^{i\omega \sum_{j=0}^{q-1} \vartheta_j} \right\rangle. \quad (10)$$

Assuming that the joint probability $P(\vartheta_0, \vartheta_1, \dots, \vartheta_{q-1})$ exist we can write the average in the above equation as

$$\left\langle e^{i\omega \sum_{j=0}^{q-1} \vartheta_j} \right\rangle = \int d\vartheta_0 \int d\vartheta_1 \dots \int d\vartheta_{q-1} P(\vartheta_0, \vartheta_1, \dots, \vartheta_{q-1}) e^{i\omega \sum_{j=0}^{q-1} \vartheta_j} \quad (11)$$

$$\begin{aligned} &= \int d\vartheta_0 P_\vartheta(\vartheta_0) e^{i\omega \vartheta_0} \int d\vartheta_1 P(\vartheta_1 | \vartheta_0) e^{i\omega \vartheta_1} \dots \\ &\quad \times \int d\vartheta_{q-1} P(\vartheta_{q-1} | \vartheta_0, \vartheta_1, \dots, \vartheta_{q-2}) e^{i\omega \vartheta_{q-1}}. \end{aligned} \quad (12)$$

If the inter-pulse durations follow the Markov process then the conditional probabilities depend only on the previous value of the inter-pulse duration, $P(\vartheta_j | \vartheta_0, \vartheta_1, \dots, \vartheta_{j-1}) = P(\vartheta_j | \vartheta_{j-1})$. In this case

$$\begin{aligned} \left\langle e^{i\omega \sum_{j=0}^{q-1} \vartheta_j} \right\rangle &= \int d\vartheta_0 P_\vartheta(\vartheta_0) e^{i\omega\vartheta_0} \int d\vartheta_1 P(\vartheta_1|\vartheta_0) e^{i\omega\vartheta_1} \dots \\ &\times \int d\vartheta_{q-1} P(\vartheta_{q-1}|\vartheta_{q-2}) e^{i\omega\vartheta_{q-1}}. \end{aligned} \quad (13)$$

Let us consider a situation when the probability density function (PDF) of the inter-pulse durations $P_\vartheta(\vartheta)$ is significant only for ϑ in some range $\vartheta_{\min} \leq \vartheta \leq \vartheta_{\max}$ and is very small for ϑ outside this range. In addition, we will assume that the conditional probability $P(\vartheta_j|\vartheta_{j-1})$ has the following properties: the average is equal to the previous value of inter-pulse duration

$$\int P(\vartheta_j|\vartheta_{j-1}) \vartheta_j d\vartheta_j = \vartheta_{j-1} \quad (14)$$

and the dispersion

$$\sigma^2 = \int P(\vartheta_j|\vartheta_{j-1}) (\vartheta_j - \vartheta_{j-1})^2 d\vartheta_j \quad (15)$$

is much smaller than the dispersion of the inter-pulse durations

$$\sigma_\vartheta^2 = \int P_\vartheta(\vartheta) (\vartheta - \bar{\vartheta})^2 d\vartheta. \quad (16)$$

These assumptions denote that the average difference between the neighbouring inter-pulse durations is small, i.e. the increments and decrements of the inter-event duration are small in comparison to the inter-event time itself.

When $\vartheta_{\max} \gg \vartheta_{\min}$ then the dispersion of the inter-pulse durations is $\sigma_\vartheta^2 \sim \vartheta_{\max}^2$. Thus, we assume that $\sigma \ll \vartheta_{\max}$. When the assumptions (14) and $\sigma \ll \sigma_\vartheta$ hold, we can approximate the conditional probability $P(\vartheta_j|\vartheta_{j-1})$ by a δ -function: $P(\vartheta_j|\vartheta_{j-1}) \approx \delta(\vartheta_j - \vartheta_{j-1})$. The approximation in equation (13) is valid only for sufficiently small q , smaller than some maximum value q_{\max} , because the error grows with the number of terms. Using in equation (13) the approximation of the conditional probability by δ -function we obtain

$$\left\langle e^{i\omega \sum_{j=0}^{q-1} \vartheta_j} \right\rangle \approx \int_0^\infty P_\vartheta(\vartheta_0) e^{i\omega q \vartheta_0} d\vartheta_0 = \chi_\vartheta(\omega q), \quad (17)$$

where

$$\chi_\vartheta(\omega) = \int_0^\infty P_\vartheta(\vartheta) e^{i\omega\vartheta} d\vartheta \quad (18)$$

is the characteristic function of the inter-pulse durations.

We can estimate the value of q_{\max} as follows: the approximation of the conditional probability $P(\vartheta_j|\vartheta_{j-1})$ by δ -function is not applicable when the dispersion of ϑ_{q-1} for a given ϑ_0 becomes comparable with the dispersion σ_ϑ^2 . Assuming that the dispersion of ϑ_j , for a given ϑ_0 , grows linearly with j (as would be the case for a random walk) we require that $\sigma^2 q_{\max} \lesssim \sigma_\vartheta^2$ and, therefore,

$$q_{\max} \sim \frac{\vartheta_{\max}^2}{\sigma^2}. \quad (19)$$

For high enough frequency, when

$$\omega q_{\max} \vartheta_{\max} \gg 1, \tag{20}$$

the characteristic functions $\chi_{\vartheta}(\omega q)$ corresponding to large $q \sim q_{\max}$ are small and we can neglect in equation (10) the terms with $q > q_{\max}$. Including only the terms with $q \leq q_{\max}$ we get the expression for the PSD

$$S(f) \approx 2\nu \sum_{q=-q_{\max}}^{q_{\max}} \chi_{\vartheta}(\omega q). \tag{21}$$

After the summation in equation (21) we obtain

$$S(f) \approx 2\nu \int_0^{\infty} \frac{\sin\left(\left(\frac{1}{2} + q_{\max}\right)\omega\vartheta\right)}{\sin\left(\frac{\omega\vartheta}{2}\right)} P_{\vartheta}(\vartheta) d\vartheta \approx \frac{4\nu}{\omega} \int_{\omega\vartheta_{\min}}^{\omega\vartheta_{\max}} \frac{\sin(q_{\max}u)}{u} P_{\vartheta}\left(\frac{u}{\omega}\right) du. \tag{22}$$

We have dropped 1/2 in $\sin(\cdot)$ because q_{\max} is large, $q_{\max} \gg 1$. In addition, for small frequencies $\omega\vartheta_{\max} \ll 1$ we approximated $\sin(u/2)$ in the denominator as $u/2$. The function $\sin(q_{\max}u)/u$ has a sharp peak of the width π/q_{\max} at $u = 0$ and decreases at larger u . If $\omega\vartheta_{\max} \gg \pi/q_{\max}$ then this peak is much narrower than the width of the PDF P_{ϑ} . In addition, the peak of the function $\sin(q_{\max}u)/u$ has a significant overlap with P_{ϑ} when $\omega\vartheta_{\min} \ll \pi/q_{\max}$. In this case we obtain the following approximate expression for the PSD:

$$S(f) \approx \frac{4\nu}{\omega} P_{\vartheta}(\vartheta_{\min}) \int_0^{\infty} \frac{\sin(q_{\max}u)}{u} du = \frac{\nu}{f} P_{\vartheta}(\vartheta_{\min}). \tag{23}$$

This equation shows that we get the 1/f spectrum.

Summing up the assumptions made above, the range of the frequencies where this expression for the PSD holds is

$$\frac{\sigma^2}{\vartheta_{\max}^3} \ll f \ll \min\left(\frac{\sigma^2}{\vartheta_{\min}\vartheta_{\max}^2}, \frac{1}{\vartheta_{\max}}\right). \tag{24}$$

When $\vartheta_{\min} < \sigma^2/\vartheta_{\max}$ the upper limit of the frequency range is determined by ϑ_{\max} . In this case the ratio of the upper and lower limiting frequencies is $\vartheta_{\max}^2/\sigma^2$. For larger ϑ_{\min} the ratio of the upper and lower limiting frequencies is $\vartheta_{\max}/\vartheta_{\min}$.

As an example, let us consider the point process where the inter-pulse durations perform a random walk and are related via the equation

$$\vartheta_{j+1} = \vartheta_j \pm \sigma. \tag{25}$$

Here each sign occurs with probability 1/2. In addition, we have reflections from the minimum inter-pulse duration $\vartheta_{\min} = 0$ and from the maximum inter-pulse duration ϑ_{\max} . The numerically obtained PSD of this signal is shown in figure 2. We see a power-law part in the PSD with the slope -1 in a broad range of frequencies from $4. \times 10^{-5}$ to 10^{-1} . This range of frequencies agrees with the estimation (24).

The PSD of the power-law form with the exponents different from -1 can be obtained by including in equation (14) an additional drift term. In [16] it was shown that the drift term of the power-law form ϑ^{δ} and power-law PDF of the inter-pulse

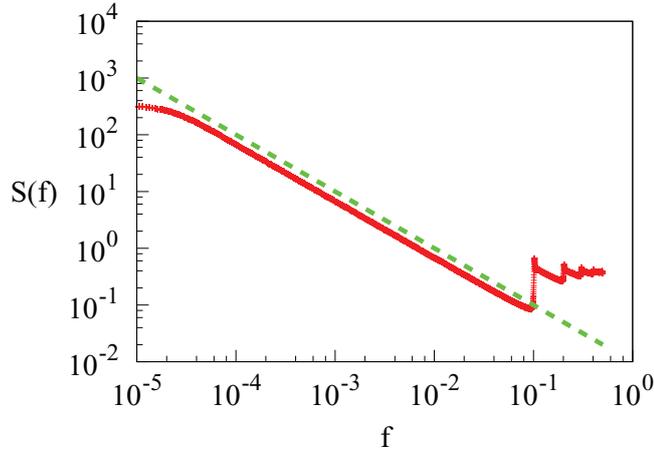


Figure 2. The PSD of a signal when the inter-pulse duration performs a random walk (25). The dashed (green) line shows $1/f$ spectrum. The parameters used are $\vartheta_{\min} = 0$, $\vartheta_{\max} = 10$, $\sigma = 0.1$.

duration $P_{\vartheta}(\vartheta) \sim \vartheta^{\alpha}$ lead to the power-law PSD $S(f) \sim 1/f^{\beta}$ with $\beta = 1 + \alpha/(2 - \delta)$. As a process generating the power-law probability distribution function for ϑ_j a multiplicative stochastic process

$$\vartheta_{j+1} = \vartheta_j + \gamma \vartheta_j^{2\mu-1} + \sigma \vartheta_j^{\mu} \varepsilon_j \quad (26)$$

has been suggested. Here ε_k are normally distributed uncorrelated random variables with a zero expectation and unit variance. For this process $\delta = 2\mu - 1$ and $\alpha = 2\gamma/\sigma^2 - 2\mu$. Equation (26) has been used for modelling the inter-note interval sequences of musical rhythms [44].

3. Time-subordinated Langevin equations

In this section we generalise the model presented in the previous section. We do this by noticing that in the pulse sequence there are two strictly increasing sequences of numbers: the physical time t and the pulse number k . The pulse number can be interpreted as an internal time of the pulse sequence. The relation between the physical time and the internal time is not deterministic because the inter-pulse durations are random. Thus, we propose the introduction of the difference between the physical and the internal operational time as a way to obtain $1/f$ noise also for other stochastic processes. To do this we start with a stochastic process and interpret the time as an internal parameter. In addition to this stochastic process we need to include an additional relation between the physical time and the internal time. In order to maintain a similarity to the point process described in the previous section, the increments of the physical time should be a power-law function of the magnitude of the signal. In this section as an initial stochastic process we take a process described by a stochastic differential equation.

A Langevin equation coupled to an additional equation for the physical time has been introduced to describe the anomalous diffusion [37, 38]. In particular, a position-dependent time subordinator was investigated in [45].

Let us start with the Langevin equation describing the diffusion of the particle subjected to an external force

$$dx_t = a(x_t)dt + b(x_t)dW_t. \tag{27}$$

Here $a(x)$ and $b(x)$ are the drift and diffusion coefficients and W_t is a standard Wiener process. For generality we assume that both coefficients a and b can depend on the stochastic variable x . In a case when the diffusion coefficient b in equation (27) depends on x we assume Itô interpretation. In equation (27) we replace the physical time t by the operational time τ ,

$$dx_\tau = a(x_\tau)d\tau + b(x_\tau)dW_\tau. \tag{28}$$

The PDF $P_x(x; \tau)$ of the stochastic variable x as a function of the operational time τ obeys the Fokker–Planck equation corresponding to Itô SDE (28) [46]

$$\frac{\partial}{\partial \tau} P_x(x; \tau) = -\frac{\partial}{\partial x} a(x)P_x(x; \tau) + \frac{1}{2} \frac{\partial^2}{\partial x^2} b^2(x)P_x(x; \tau). \tag{29}$$

Primarily we consider the situation when the small increments of the physical time are deterministic and are proportional to the increments of the operational time. Thus, the physical time t is related to the operational time τ via the equation

$$dt_\tau = g(x_\tau)d\tau. \tag{30}$$

Here the positive function $g(x)$ is the intensity of random time that depends on the intensity of the signal x . If we interpret equation (27) as describing the diffusion of a particle in a non-homogeneous medium, function $g(x)$ models the position of the structures responsible for either trapping or accelerating the particle [45]. Large values of $g(x)$ corresponds to trapping the particle, whereas small $g(x)$ leads to the acceleration of diffusion. For fixed particle position x the coefficient $g(x)$ in equation (30) is constant and from equation (30) follows the relationship

$$\frac{\partial}{\partial \tau} P(t; \tau|x) = -\frac{\partial}{\partial t} g(x)P(t; \tau|x). \tag{31}$$

for the PDF $P(t; \tau|x)$ of the physical time t as a function of the operational time τ . Equations (28) and (30) together define the subordinated process. However, now the processes $x(\tau)$ and $t(\tau)$ are not independent.

Let us derive the Langevin equation for the stochastic variable x in the physical time t . To do this, we consider the joint PDF $P_{x,t}(x, t; \tau)$ of the stochastic variables x and t . Equations (28) and (30) yield the two-dimensional Fokker–Planck equation

$$\frac{\partial}{\partial \tau} P_{x,t}(x, t; \tau) = -\frac{\partial}{\partial x} a(x)P_{x,t} + \frac{1}{2} \frac{\partial^2}{\partial x^2} b^2(x)P_{x,t} - \frac{\partial}{\partial t} g(x)P_{x,t}. \tag{32}$$

This equation is a combination of equations (29) and (31). The zero of the physical time t coincides with the zero of the operational time τ , therefore, the initial condition for equation (32) is $P_{x,t}(x, t; 0) = P_x(x, 0)\delta(t)$. Coinciding zeros of t and τ also lead to the boundary condition $P_{x,t}(x, 0; \tau) = 0$ for $\tau > 0$, because t and τ are strictly increasing.

Instead of x and t we can consider x and τ as stochastic variables. The stochastic variable t is related to the operational time τ via equation (30), therefore, the joint PDF

$P_{x,\tau}(x, \tau; t)$ of the stochastic variables x and τ is related to the PDF $P_{x,t}(x, t; \tau)$ according to the equation

$$P_{x,\tau}(x, \tau; t) = g(x)P_{x,t}(x, t; \tau). \quad (33)$$

This equation can be obtained by noticing that the last term in equation (32) contains derivative $\frac{\partial}{\partial t}$ and thus should be equal to $-\frac{\partial}{\partial t}P_{x,\tau}$. Using equations (32) and (33) we get

$$\frac{\partial}{\partial t}P_{x,\tau}(x, \tau; t) = -\frac{\partial}{\partial x}a(x)\frac{1}{g(x)}P_{x,\tau} + \frac{1}{2}\frac{\partial^2}{\partial x^2}b^2(x)\frac{1}{g(x)}P_{x,\tau} - \frac{\partial}{\partial \tau}\frac{1}{g(x)}P_{x,\tau}. \quad (34)$$

The PDF $P_{x,\tau}$ has the initial condition $P_{x,\tau}(x, \tau; 0) = P_x(x, 0)\delta(\tau)$ and the boundary condition $P_{x,\tau}(x, 0; t) = 0$ for $t > 0$. The PDF of the subordinated random process x_t is $P(x, t) = \int P_{x,\tau}(x, \tau; t)d\tau$. Integrating both sides of equation (34) we obtain

$$\frac{\partial}{\partial t}P(x, t) = -\frac{\partial}{\partial x}\frac{a(x)}{g(x)}P(x, t) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\frac{b^2(x)}{g(x)}P(x, t). \quad (35)$$

Thus, position-dependent trapping leads to the position-dependent coefficients in the Fokker–Planck equation, even if the initial SDE (28) has constant coefficients. Equation (35) corresponds to the single equation in the physical time with the multiplicative noise,

$$dx_t = \frac{a(x_t)}{g(x_t)}dt + \frac{b(x_t)}{\sqrt{g(x_t)}}dW_t. \quad (36)$$

In fact, the Fokker–Planck equation (34) can be obtained from the coupled equations (36) and

$$d\tau_t = \frac{1}{g(x_t)}dt. \quad (37)$$

The relationship between the physical time t and the operational time τ cannot be necessarily deterministic, equation (30) can have a stochastic term. If the fluctuations of this stochastic term are much faster than the fluctuations of the stochastic variable x , we can approximate them by the average value. In this case $g(x)$ describes the average increment of the physical time. If this average is positive, the derivation presented above is still valid and equation (36) holds.

4. Example equations generating signals with 1/f noise

In this section we consider several stochastic processes and, introducing the internal and external times, we check whether 1/f noise can be obtained. In a signal consisting of pulses the internal time is just the pulse number and the increment of the physical time is equal to the inter-pulse duration. The intensity of this signal is inversely proportional to the inter-pulse duration. In order to obtain 1/f noise similarly as for the signal

consisting of pulses we choose function $g(x)$ in equation (30) as a power-law function of x , $g(x) \sim x^{-2\eta}$, where η is the power-law exponent.

Let us start from a simple Brownian motion

$$dx_\tau = dW_\tau. \quad (38)$$

In order to keep the stochastic variable x always positive we include reflective boundary at $x = x_{\min} > 0$. We consider equation (38) together with the relation

$$dt_\tau = x_\tau^{-2\eta} d\tau \quad (39)$$

between the physical time t and internal time τ . According to (36) the resulting equation in the physical time is

$$dx_t = x_t^\eta dW_t. \quad (40)$$

More generally, the initial equation can include a position-dependent force. If we take the equation describing the Bessel process

$$dx_\tau = \left(\eta - \frac{\lambda}{2} \right) \frac{1}{x_\tau} d\tau + dW_\tau \quad (41)$$

together with equation (39), then the resulting equation in physical time becomes

$$dx_t = \left(\eta - \frac{\lambda}{2} \right) x_t^{2\eta-1} dt + x_t^\eta dW_t. \quad (42)$$

Here the parameter λ gives the power-law exponent of the steady-state PDF. The same equation (42) in physical time arises starting from the geometric Brownian motion,

$$dx_\tau = \left(\eta - \frac{\lambda}{2} \right) x_\tau d\tau + x_\tau dW_\tau, \quad (43)$$

and the relation between the internal time and physical time

$$dt_\tau = x_\tau^{-2(\eta-1)} d\tau. \quad (44)$$

Nonlinear SDE (42) for generating signals with $1/f^\beta$ spectrum was proposed in [32, 33]. As was shown in [47], the reason for the appearance of the $1/f$ spectrum is the scaling properties of the signal: the change in the magnitude of the variable $x \rightarrow ax$ is equivalent to the change in the time scale $t \rightarrow a^{2(\eta-1)}t$. The connection of the power-law exponent β in the PSD with the parameters of equation (42) is given by the equation [33, 47]

$$\beta = 1 + \frac{\lambda - 3}{2(\eta - 1)}. \quad (45)$$

Analysis [48] of SDE (42) shows that equation (45) is valid only for the values of the parameters η and λ yielding $0 < \beta < 2$.

The nonlinear SDE (42) leads to the stationary process and non-diverging steady state PDF only when the diffusion of stochastic variable x is restricted. The simplest choice of the restriction is the reflective boundary conditions at $x = x_{\min}$ and $x = x_{\max}$. The presence of the restrictions of diffusion makes the scaling properties of equation (42)

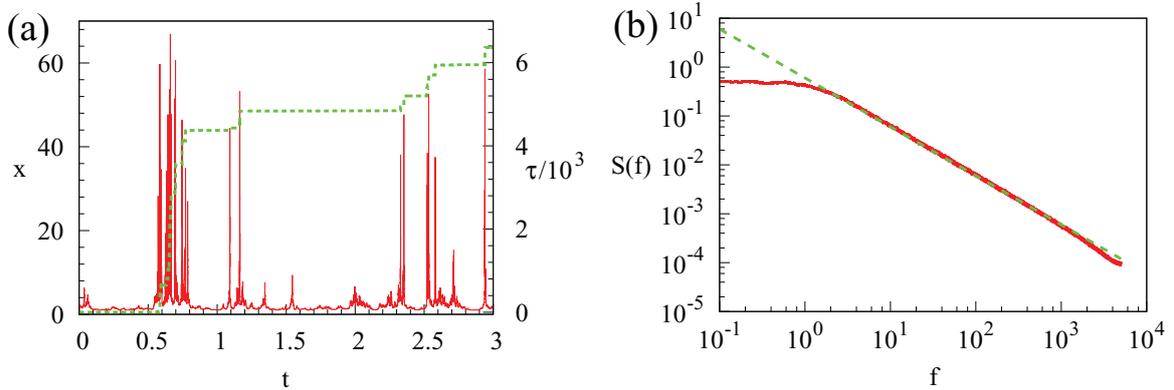


Figure 3. (a) Signal generated by equation (42) with the parameters $\eta = 5/2$ and $\lambda = 3$ (solid red line) together with the corresponding internal time (dashed green line). Reflective boundaries at $x_{\min} = 1$ and $x_{\max} = 1000$ have been used. (b) PSD of the generate signal. The dashed green line shows the slope $1/f$.

only approximate and limits the power-law part of the PSD to a finite range of frequencies. This range of frequencies has been qualitatively estimated as [47]

$$\begin{aligned} x_{\min}^{2(\eta-1)} &\ll 2\pi f \ll x_{\max}^{2(\eta-1)}, & \eta > 1, \\ x_{\max}^{-2(1-\eta)} &\ll 2\pi f \ll x_{\min}^{-2(1-\eta)}, & \eta < 1. \end{aligned} \quad (46)$$

By increasing the ratio x_{\max}/x_{\min} one can get an arbitrarily wide range of the frequencies where the PSD has $1/f^\beta$ behaviour.

An example of a signal generated by equation (42) together with the internal time τ is shown in figure 3(a). We used the parameters $\eta = 5/2$, $\lambda = 3$ and reflective boundaries at $x_{\min} = 1$ and $x_{\max} = 1000$. The method of the numerical solution is discussed in the next section. We see that internal time τ increases rapidly when the signal x acquires large values and τ changes slowly when x is small. According to equation (45) the choice of $\lambda = 3$ should result in $1/f$ behaviour of the PSD. The corresponding power spectral density $S(f)$ is shown in figure 3(b). The numerical solution of the equation confirms a presence of a wide region of frequencies where the spectrum has $1/f$ behaviour.

When the stochastic variable x can acquire both positive and negative values, the function $g(x)$ cannot be just a simple power-law, because $g(x)$ becomes unbounded or equal to zero when $x \rightarrow 0$. In order to avoid this problem we require that function $g(x)$ should have power-law behaviour only asymptotically, for large values of $|x|$. One of the possible choices is

$$g(x) = \frac{1}{(x^2 + x_0^2)^\eta}. \quad (47)$$

Here we added a constant x_0 that corrects the behaviour of function $g(x)$ at $x = 0$. The power-law behaviour is preserved when $|x| \gg x_0$.

The stochastic variable x can acquire both positive and negative values if we start from the Ornstein–Uhlenbeck process

$$dx_\tau = -\gamma x_\tau d\tau + dW_\tau. \quad (48)$$

Here the parameter γ is the relaxation rate. We consider equation (48) together with the relation

$$dt_\tau = \frac{1}{(x_\tau^2 + x_0^2)^\eta} d\tau \quad (49)$$

between the physical time t and internal time τ . According to (36), equations (48) and (49) leads to SDE

$$dx_t = -\gamma(x_t^2 + x_0^2)^\eta x_t dt + (x_t^2 + x_0^2)^{\frac{\eta}{2}} dW_t \quad (50)$$

in the physical time t . Equation (50) can be written as

$$dx_t = \left(-\frac{x_t^2 + x_0^2}{x_{\max}^2} \right) (x_t^2 + x_0^2)^{\eta-1} x_t dt + (x_t^2 + x_0^2)^{\frac{\eta}{2}} dW_t \quad (51)$$

where

$$x_{\max} = \frac{1}{\sqrt{\gamma}} \quad (52)$$

defines a cut-off position at large values of x .

Another interesting equation describing the evolution in internal time is

$$dx_\tau = \left(\eta - \frac{\lambda}{2} \right) \frac{x_\tau}{x_\tau^2 + x_0^2} d\tau + dW_\tau. \quad (53)$$

In this equation the relaxation rate depends on the magnitude of the signal. If $|x| \ll x_0$ we get the equation of Ornstein–Uhlenbeck type, whereas for large values of $|x|$ the relaxation decreases with increasing $|x|$. Equation (53) together with (49) result in the following equation in the physical time:

$$dx_t = \left(\eta - \frac{\lambda}{2} \right) (x_t^2 + x_0^2)^{\eta-1} x_t dt + (x_t^2 + x_0^2)^{\frac{\eta}{2}} dW_t. \quad (54)$$

Finally, the combination of equations (48) and (53),

$$dx_\tau = -\left(\gamma - \left(\eta - \frac{\nu}{2} \right) \frac{1}{x_\tau^2 + x_0^2} \right) x_\tau d\tau + dW_\tau, \quad (55)$$

together with (49) leads to a more general equation in the physical time

$$dx_t = \left(\eta - \frac{\nu}{2} - \frac{x_t^2 + x_0^2}{x_{\max}^2} \right) (x_t^2 + x_0^2)^{\eta-1} x_t dt + (x_t^2 + x_0^2)^{\frac{\eta}{2}} dW_t. \quad (56)$$

The nonlinear SDE (54) was investigated in [49]. It was shown that SDE (54) generates a signal with the steady-state PDF described by the q -Gaussian distribution featuring in the non-extensive statistical mechanics. In addition, the spectrum of the generated signal has $1/f^\beta$ behaviour in a wide range of frequencies, with the power-law exponent β given by equation (45).

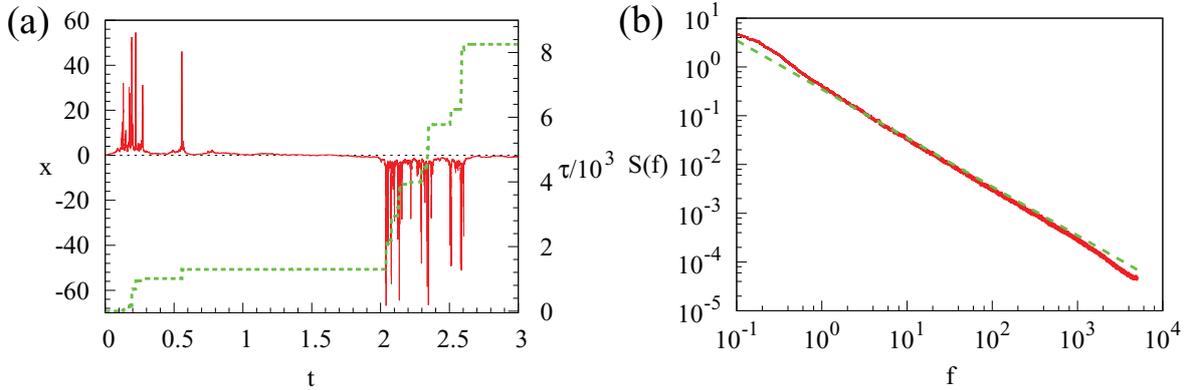


Figure 4. (a) Signal generated by equation (54) with the parameters $\eta = 5/2$, $\lambda = 3$ and $x_0 = 1$ (solid red line) together with the corresponding internal time (dashed green line). (b) PSD of the generate signal. The dashed green line shows the slope $1/f$.

An example of a signal generated by equation (54) together with the internal time is shown in figure 4(a). We used the parameters $\eta = 5/2$, $\lambda = 3$ and $x_0 = 1$. We see that the internal time τ increases rapidly when the absolute value of the signal x is large and τ changes slowly when the absolute value of x is small. The internal time τ increases both for positive and negative values of x . The PSD of a signal generated by equation (54) is shown in figure 4(b). The numerical solution confirms a presence of a region where the spectrum behaves as $1/f$. Thus, the introduction of negative values of x does not destroy $1/f$ spectrum.

5. Numerical approach

Introduction of the internal time can be an effective technique for the solution of highly non-linear SDEs. For the numerical solution of nonlinear equations the solution schemes involving a fixed time step Δt can be inefficient. For example, in equation (42) with $\eta > 1$ large values of stochastic variable x lead to large coefficients and thus require a very small time step. The numerical solution scheme can be improved by introducing the internal time τ that is different from the real physical time t .

Let us consider equation (42) with the noise multiplicativity exponent $\eta > 1$. We can introduce internal time τ using the equation

$$d\tau_t = x_t^{2\eta} dt. \tag{57}$$

Then, according to equations (36) and (37), SDE (42) is equivalent to coupled equations

$$dx_\tau = \left(\eta - \frac{\nu}{2} \right) \frac{1}{x_\tau} d\tau + dW_\tau, \tag{58}$$

$$dt_\tau = \frac{1}{x_\tau^{2\eta}} d\tau. \tag{59}$$

Now, equation (58) is much simpler than the initial equation (42). Discretising the internal time τ with the step $\Delta\tau$ and using the Euler–Marujama approximation for the SDE (58) we get

$$x_{k+1} = x_k + \left(\eta - \frac{\lambda}{2}\right) \frac{1}{x_k} \Delta\tau + \sqrt{\Delta\tau} \varepsilon_k, \tag{60}$$

$$t_{k+1} = t_k + \frac{\Delta\tau}{x_k^{2\eta}}. \tag{61}$$

Here ε_k are normally distributed uncorrelated random variables. Equations (60) and (61) provide the numerical method for solving equation (42). One can interpret equations (60), (61) as an Euler–Marujama scheme with a variable time step $\Delta t_k = \Delta\tau/x_k^{2\eta}$ that adapts to the coefficients in the equation. The cost of the introduction of the internal time is the randomness of the increments of the real physical time t . To get the discretisation of time with fixed steps the signal generated in such a way should be interpolated.

Another possible choice is to introduce the internal time τ by the equation

$$d\tau_t = x_t^{2(\eta-1)} dt. \tag{62}$$

In this case we obtain a different pair of equations

$$dx_\tau = \left(\eta - \frac{\lambda}{2}\right) x_\tau d\tau + x_\tau dW_\tau, \tag{63}$$

$$dt_\tau = \frac{1}{x_\tau^{2(\eta-1)}} d\tau. \tag{64}$$

Note that now the internal time τ is dimensionless even if x and t are not. Discretising the internal time τ with the step $\Delta\tau$ and using the Euler–Marujama approximation for the SDE (63) we obtain

$$x_{k+1} = x_k + \left(\eta - \frac{\lambda}{2}\right) x_k \Delta\tau + x_k \sqrt{\Delta\tau} \varepsilon_k. \tag{65}$$

$$t_{k+1} = t_k + \frac{\Delta\tau}{x_k^{2(\eta-1)}}. \tag{66}$$

This method of solution was proposed in [32]. On the other hand, using the Milstein approximation for the SDE (63) we have

$$x_{k+1} = x_k + \left(\eta - \frac{\lambda}{2}\right) x_k \Delta\tau + x_k \sqrt{\Delta\tau} \varepsilon_k + \frac{1}{2} x_k \Delta\tau (\varepsilon_k^2 - 1), \tag{67}$$

$$t_{k+1} = t_k + \frac{\Delta\tau}{x_k^{2(\eta-1)}}. \tag{68}$$

Note that the last term in equation (67) differs from the corresponding term in the equation obtained just by using a variable time step $\Delta t = \Delta\tau/x_k^{2(\eta-1)}$ in the Milstein approximation for equation (42).

A numerical simulation of subordinated equations using fixed steps of operational time and random increments of physical time was discussed in [42, 43]. A variable time step makes the numerical simulation in [42, 43] similar to the method proposed in this section. The main difference of our method from previous discussions of subordinated equations lies in the dependence of the increment of the physical time on the magnitude of the signal x .

6. Discussion and conclusions

In summary, we have demonstrated that starting from a random process described by a SDE and introducing the difference between the internal time and the physical time $1/f$ behaviour of the PSD can be obtained.

One of the physical situations where the difference between the internal and physical time can arise is transport in an inhomogeneous medium. Impurities and regular structures in the medium can cause transport of variable speed, the particle may be trapped for some time or accelerated. Nonhomogeneous systems exhibit not only subdiffusion related to traps, but also enhanced diffusion as a result of the disorder. For example, the movement of particles between two neighbouring lattice sites in an interacting particle system is superdiffusive due to disorder and subdiffusive without disorder [50]. The dynamics in a medium with traps is described by the continuous time random walk theory (CTRW) [51, 52]. In a description equivalent to the CTRW the dynamics of the particle is Markovian and governed by the Langevin equation in an auxiliary operational time instead of physical time. This Markovian process is subordinated to the process yielding physical time.

In the case of subdiffusion the PSD of the signals generated by subordinated Langevin equations has power-law behaviour $S(f) \sim f^{\alpha-1}$ as $f \rightarrow 0$ [53], where α is the power-law exponent in the time dependence of the mean square displacement. Since for subdiffusion $0 < \alpha < 1$, the power-law exponent β in the PSD is smaller than 1. The results obtained in this paper suggest that $1/f$ noise in subdiffusion should occur in a heterogeneous medium, where the trapping time depends on the position [54].

The traditional CTRW provides a homogeneous description of the medium. A more complex situation is the diffusion in nonhomogeneous media, for example, diffusion on fractals and multifractals [55]. A heterogeneous medium with steep gradients of diffusivity can be created via a local variation of the temperature in thermophoresis experiments [56, 57]. Spatial heterogeneities are also present in the case of anomalous diffusion in subsurface hydrology [58]. In the random walk description spatially varying diffusivity can be translated into a local dependence of the waiting time for a jump event. In the heterogeneous medium the properties of a trap can reflect the medium structure, thus in the description of transport the waiting time should explicitly depend on the position of the particle [45]. A method to include position dependent waiting time is a consideration of the position-dependent time subordinator [45].

In general, the trapping time can depend not on the position of the particle but on some other quantity. Then in the dynamics of this quantity the difference between the physical and operational time also arises, with the relationship between the times dependent on the intensity of the signal.

In socio-economical systems the internal time can reflect fluctuating human activity [35]. For example, in finance the long-range correlations in volatility arise due to fluctuations in the trading activity [59, 60].

We have shown that 1/f noise occurs when the internal time and physical time are related via the power-law function of the signal intensity, for example, via equations (39) or (49). Although we have considered only random processes described by a SDE, we expect that the mechanism of the appearance of 1/f noise presented here is quite general and should also work for other random processes. We anticipate that the present model can be useful for explaining 1/f noise in different complex systems.

In addition, we suggested a way of solving highly non-linear SDEs by introducing suitably chosen internal time and variable steps of integration.

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