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INSTITUTE OF THEORETICAL PHYSICS AND ASTRONOMY

Darius Jurčiukonis

CANONICAL QUANTIZATION OF SU(3) TOPOLOGICAL SOLITONS

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Scientific supervisor

Prof. Dr. Egidijus NORVAIŠAS, (Vilnius University Institute of Theoretical Physics and Astronomy, Physical Sciences, Physics – 02 P, mathematical and general theoretical physics, classical mechanics, quantum mechanics, relativity, gravitation, statistical physics, thermodynamics – 190 P)

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Darius Jurčiukonis

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Mokslinis vadovas

prof. dr. Egidijus NORVAIŠAS (Teorinės fizikos ir astronomijos institutas, fiziniai mokslai, fizika – 02 P, matematinė ir bendroji teorinė fizika, klasikinė mechanika, kvantinė mechanika, reliatyvizmas, gravitacija, statistinė fizika, termodinamika – 190 P)

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Notations and conventions

- Bold letters indicate multiple quantity structure, which may vary from case to case. For example, $\boldsymbol{\tau}$ denotes a triple of Pauli matrices (τ_1, τ_2, τ_3), $\mathbf{r}$ denotes the spatial vector $\vec{r}$ and $\mathbf{U}$ denotes the unitary field. Note that sometimes we do not follow this convention for the coordinate q, momentum p and the variables α or $\varkappa$ in order to keep expressions more compact.

- Calligraphic letters like $\mathcal{L}, \mathcal{M}, \mathcal{V}, \mathcal{H}$, etc. (with the exception of $\mathcal{I}$ and $\mathcal{O}$) denote densities.

- The appearance of carets ($\hat{R}$, $\hat{L}$, $\hat{J}$, etc.) indicate an operator or its component. Note that we do not follow this convention for the coordinate q and momentum p operators in order to keep notations simpler. This is also true for operators which are functions of q only.

- The dot over a symbol ($\dot{q}$, $\dot{\alpha}$, etc.) denotes the full time derivative.

- The metric tensor $g^{\mu\nu}$ is $g^{00} = 1, g^{0,i} = 0, g^{i,j} = -\delta^{i,j}$ for spatial indices $i, j = 1, 2, 3$. The derivative $\partial^\mu \equiv \frac{\partial}{\partial x_\mu}$ has components $(\partial/\partial t, -\nabla)$. The sign of totally anti-symmetric tensors (Levi-Cevita symbols) ϵ_{ijk} and $\epsilon^{\mu\nu\sigma\gamma}$ are fixed by $\epsilon_{123} = -\epsilon^{123} = 1$ and $\epsilon_{0123} = -\epsilon^{0123} = 1$, respectively.

- The isovector of Pauli isospin matrices $\boldsymbol{\tau}$ in Cartesian coordinates have a form $\tau_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\tau_2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$, $\quad \tau_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

- The symbol $*$ is a complex conjugation mark.

- The cross over some operators $\overset{+}{A}$ denotes hermitically conjugate functions and is equivalent to notation $A^\dagger$.

- SU(2), SU(3), SO(3), SU(N), etc. denotes the symmetry groups.

- Greek letters $\alpha, \beta, \gamma \ldots$ are the summation indices when used as upper indices. They also represent Euler angles. The middle Greek letters λ, μ indicate representation of the SU(3) group, whereas θ, φ represent angles of the spherical coordinates. The Latin letters $a, b, c, d \ldots$ are the summation indices usually. Minuscule letters $j, l, m, n \ldots$ indicate parameters of SU(2) group, whereas capital letters $L, I, M, Z \ldots$ represent parameters of SU(3) or SO(3) groups. The index t denotes the time component.

- The symbol $\mathbb{1}$ denotes the unit matrix.

- The curly brackets $\{\ ,\ \}$ and the square brackets $[\ ,\ ]$ denote the anti-commutator and the commutator, respectively.

- There is the assumed summation convention under the repeated (dummy) indices. The symbol $\sum\limits^{(\lambda,\mu)}$ indicates summation over SU(2) group representations, which are included in (λ,μ) irreducible representation.

List of publications

1. D. Jurčiukonis and E. Norvaišas, Quantum SU(3) Skyrme model with noncanonical embedded SO(3) soliton, J. Math. Phys., **48**, 052101 (2007).

2. D. Jurčiukonis and E. Norvaišas, Quantum SU(3) Skyrme model for arbitrary representation, Bulg. J. Phys., **33** (s1b), 933 (2006).

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- "IV. International Symposium Quantum Theory and Symmetries (QTS-4)", Varna, August 15–21, 2005.
- "6$^{\text{th}}$ Nordic Summer School in Nuclear Physics", Copenhagen, August 8–19, 2005.
- "36$^{\text{th}}$ Lithuanian National Conference of Physics", Vilnius, June 16–18, 2005.
- "Physics in Warsaw 2004", Summer School, Warsaw, September 20 – October 02, 2004.
- "Conference on Computational Physics 2004 (CCP-2004)", Genoa, September 01–04, 2004.
- "35$^{\text{th}}$ Lithuanian National Conference of Physics", Vilnius, June 12–14, 2003.

Preface

The first suggestion that the ordinary proton and neutron might be viewed as topological solitons was made by British physicist T.H.R. Skyrme in the sixties of last century. In his papers [1, 2] he has described nucleons as a liquid of pions by using a Lagrangian in which only two phenomenological constants were incorporated. The technology of the quantum field theory in the sixties was not sufficiently advanced to treat solitons and it took almost twenty years before the ideas of Skyrme were revived by Witten [3] and Balachandran *et al.* [4]. The first demonstration that the Skyrme model could fit the observed properties of the baryons to an accuracy of about 30% was made by Adkins *et al.* [5]. The model was subsequently refined and extended in many ways. From phenomenological applications in elementary particle and nuclear physics the Skyrme model was applied to study the quantum Hall effect [6] and Bose-Einstein condensates [7] as well as in cosmology [8].

The equations of solitons are highly nonlinear and almost always are not solvable analytically, while the direct quantization of the solitonic solutions leads to rather complicated equations. In our work we use the quantization in "zero modes" or "collective coordinate" approach [5, 9], which leads to a consistent quantum description. Using the canonical quantization we consider the Skyrme Lagrangian quantum mechanically *ab initio*. In this case the generalized coordinates and velocities do not commute. The canonical quantization leads to quantum corrections, which stabilize the soliton solutions. To obtain Euler-Lagrange equations, that are consistent with the canonical equation of motion of the Hamiltonian, the general method of quantization on a curved space developed by Sugano *et al.* [10] is employed.

The Skyrme model is usually formulated in the fundamental representation of SU(2) group, where the field is a unitary 2×2 matrix. The model can also be generalized to unitary fields, that belong to general representations of the SU(2) [11–13], along with a demonstration that the quantum corrections are representation dependent.

The quantum SU(3) Skyrme model was generalized to arbitrary representation as well [14]. The algebraic structure of the SU(3) group is more plentiful, consequently it is possible to use various classical solutions for the quantization. The choice of the noncanonically embedded soliton is explored in Ref. [15]. The formalism of the noncanonical embedded soliton can be expanded for the description of multisolitonic states by using rational map approximation. These questions are considered in this dissertation.

The main goals of the research work

1. To investigate representation dependence of the canonically quantized SU(3) Skyrme model.

2. To explore the SU(3) Skyrme model with a noncanonically embedded SU(3) $\supset$ SO(3) soliton.

3. To investigate the SU(3) Skyrme model with the noncanonically embedded SU(3) $\supset$ SO(3) soliton by using the rational map approximation with the baryon number $B \geq 2$.

Scientific novelty

This work exposes new possibilities of the extended basic SU(3) Skyrme model to general irreducible representations (λ, μ). The strict canonical quantization of the model yields from representation depended Lagrangian density, which can be treated as different solitons, considering different representations. The classical limit of these quantum Lagrangian densities is the usual SU(3) Skyrme Lagrangian density. A new, nontrivial dependence in Wess-Zumino and the symmetry breaking terms on the representation was observed.

The new ansatz for the Skyrme model is introduced. It is defined in the noncanonical SU(3) $\supset$ SO(3) bases. Quantization of the soliton leads to new expressions of the soliton momenta of inertia and quantum mass corrections. The rational map approximation for the Skyrme model with noncanonically embedded ansatz is applied. This leads to five different quantum moments of inertia and new quantum mass corrections. The explored ansatz can be used in nuclear physics to describe a light nucleus.

Generalizations considered in the work are significant and can be extended to other models and theories.

Thesis statements

1. Different Lagrangian and Hamiltonian operators of the quantum SU(3) Skyrme model for different representations (λ, μ) were found. The explicit dependence on the representation of the Wess-Zumino term, the symmetry breaking term and the quantum mass corrections were derived in the framework of the canonical quantization. The dependence on the irreducible representation (λ, μ) of the SU(3) group can be treated as a new discrete phenomenological parameter of the model.

2. A new version of the Skyrme model is obtained introducing the new ansatz, which is defined in the noncanonical SU(3) $\supset$ SO(3) base. This is confirmed by the new expressions of the soliton momenta of inertia and the quantum mass corrections.

3. The topological solitons carrying baryon number $B \geq 2$ can be described using the rational map approximation ansatz, which is a noncanonically embedded SU(3) $\supset$ SO(3) soliton. The canonical quantization leads to five different quantum moments of inertia and new quantum mass corrections.

Approbation of the results

Main results of the research described in this dissertation have been published in 3 scientific papers and a few talks in the international conferences. A detailed list of publications is given in previous section.

Personal contribution of the author

The author of the thesis performed many analytical derivations of the equations (shown in chapters II – IV and the appendices) by using "pencil" and majority calculations by using computer algebra system MATHEMATICA.

Manuscript organization

The manuscript is organized into four chapters and four appendices, containing some auxiliary expressions connected with the analyzed problems in the dissertation. The chapters have short introductions in the beginning, research of the problem and summaries at the end. It is useful to describe the purpose of each chapter. Chapter I contains the mathematical formulation and the physical motivation of the Skyrme model at the background level. This chapter contains no new results. Chapter II describes the representation dependence of the canonically quantized SU(3) Skyrmion. This chapter includes results of Ref. [14]. The formulation of the SU(3) Skyrme model with the noncanonically embedded SU(3) $\supset$ SO(3) soliton is presented in Chapter III following Ref. [15]. The rational map approximation for the Skyrme model with the noncanonically embedded ansatz is explored in Chapter IV.

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I. Introduction to Skyrme model

1. Historical remarks

While deep inelastic electron-scattering experiments at high energy show the baryons to consist of quarks, no such simplicity is revealed by the low energy observables of the baryons. In fact the low energy structure of the baryons appears to represent the full complexity of the "vacuum" of quantum chromodynamics (QCD), a description of which is entirely outside of the realm of the extant perturbative methods of QCD. On the other hand, description of the baryon structure and interactions on the basis of effective chiral meson models has been developed. The challenge has therefore been to find a sound basis in QCD for this effective mesonic description. The main task was to develop non-perturbative approaches to QCD, which could replace the basic dynamical quark and gluon degrees of freedom by observable meson fields.

When the original fermion theory (QCD) is replaced by an effective boson theory, the fermions (baryons) have to be described as soliton solutions, which are topologically stable, with a conserved integer quantum number that is to be interpreted as the baryon number [3, 16]. The only fundamental symmetry that survives the procedure of replacing the fundamental fields by the effective meson fields is the chiral symmetry of QCD. Various models were created based on the nonlinear chiral meson field theory. Their topological soliton solutions can be quantised as fermions. The simplest of these models is the Skyrme model proposed by T.H.R. Skyrme in the sixties of last century [1].

The Skyrme model is the simplest extension of the nonlinear σ-model [17] that has stable soliton solutions with an integral baryon number. It represents a theory for an interacting pion field, in which the interactions are introduced through a chirally symmetric constraint involving an auxiliary scalar meson field, σ. Although a bosonic representation of QCD should involve infinitely many meson fields, one expects only the lightest mesons to be of quantitative significance for the low energy properties of the baryons (and nuclei in general). Hence a natural starting point for the description of the low energy behaviour of the baryons is to use the Skyrme model and its immediate generalizations, which incorporate vector meson fields [18–20]. The small number of parameters of the Lagrangian model are in principle obtainable from QCD, but in practice they have to be chosen to fit some of the observables.

The technology of the quantum field theory in the sixties was not sufficiently advanced to treat solitons, so it took almost twenty years before the ideas of Skyrme were revived by Witten [3] and Balachandran *et al.* [4]. The first demonstration that the Skyrme model, in spite of its simplicity with only two parameters in the Lagrangian density, can provide a description of the static properties of the non-strange baryons was made by Adkins *et al.* [5]. The predicted observables were found to differ from their empirical values by less than 30%. The model was subsequently refined in many ways. The correct asymptotic behaviour of the soliton field was achieved by an introduction of a chiral symmetry breaking mass term [21]. Alternative and augmented forms for the basic Lagrangian density have been studied [22, 23]. In addition, different generalizations of the Skyrme model that contain explicit vector meson fields have been developed [18–20] and applied to the analysis of baryon structure [24]. In general the refinements of the topological soliton model have led to improved predictions of the low energy baryon observables, although certain systematic discrepancies remain, among them the constant underprediction of the axial-vector coupling constant.

One way of looking at these systematic quantitative shortcomings of the topological soliton model is to view the model as the zero-size limit of the chiral bag with an external meson field [25]. By replacing the central region by an explicit quark bag, which is stabilised by the external soliton field, the quark colour number N_c appears in the formulae as a parameter. Then it becomes obvious that the pure soliton model can be viewed as a large-N_c limit of the chiral bag model. In particular, it becomes clear that a factor 5/3 in the axial coupling constant is lost in the pure soliton limit, which is probably the main reason for the underprediction of this observable in the soliton model [26].

The study of the static baryon observables in the two-phase chiral bag model has also proven to be very useful because it has led to the conclusion that these observables are insensitive to the bag size [27]. This is sometimes referred to as the "Cheshire cat" principle, the essential content of which is that the low energy properties of the baryons can be described in equivalent ways in terms of quarks and gluons or meson fields, making the choice of language one of convenience rather than one of essence. In practice, the consistent treatment of the complete chiral bag model is rather subtle. Once the baryon number is divided, with a fraction carried by the soliton field, the sea quark contributions must be included as the valence quarks alone always give an integer baryon number [28]. A further complication is due to the need to rotate the soliton field and the quark bag in a consistent way for the projection onto states of good spin and isospin [27].

These and other difficulties associated with the use of the complete chiral bag model makes it natural to lean heavily on the Cheshire cat principle and to exploit all reasonable possibilities of refining the pure topological soliton model before resorting to the two-phase model. This approach has in fact been fairly successful. Using versions of the model that contain explicit vector meson fields one obtains a fairly good description not only of the static baryon observables, but of the baryon form factors as well [24].

The application of the topological soliton model to the interacting two-nucleon system was also proven to be extremely fruitful. While the need for mathematical approximations has hampered the study of the interaction operators at short distances, sufficiently accurate approximation methods have permitted the study of their long-range behaviours. Thus, many important features of the nucleon-nucleon interaction are predicted by the simplest original version of the model. Among these are the one-pion-exchange interaction [2], the short-range repulsion [26] and the isospin-dependent spin-orbit interaction [29].

The Skyrme model can also be used to study elementary particle dynamics, e.g. pion-nucleon scattering. To describe pion-nucleon scattering in the Skyrme model, an external pion field has to be coupled to the soliton field. This can be done in several formally different ways [30–32]. The simplest method is that of Schnitzer [30, 33], which leads to the usual Weinberg Lagrangian [34] for the pion-nucleon system in the quadratic approximation. Studies of pion-nucleon scattering amplitudes using the Skyrme model have led to interesting relations between different partial wave amplitudes, which appear to be satisfied empirically [31, 32].

The generalization of the Skyrme model from the SU(2) to the SU(3) flavor symmetry for a description of the strange hyperons was proven to be very interesting. The straightforward method of incorporation of strange hyperons to the model corresponds in the quark-model language to treating the strange (s) quark on the same footing as the non-strange (u, d) quarks. This leads to a description of the hyperons in terms of the SU(3) collective coordinates, in which both the kaons and the pions are massless in the first approximation. Although first attempts along these lines seemed to be successful [35, 36] but it latter appeared that the direct extension of the model, in which the isospin group is enlarged and the strangeness is treated in the same way as the isospin, does not give a quantitatively acceptable description of the hyperon spectrum [37, 38]. An alternative approach is the bound-state model due to Callan and Klebanov [39], in which the strange hyperons are described as bound states of the SU(2) solitons (skyrmions) and the strangeness-carrying kaons. This approach leads to an acceptable description of the spectrum of the stable hyperons [40, 41], and to the values of their magnetic moments that differ from the empirical ones by less than 20% [42, 43]. This model for the hyperons has intrinsic interest as well, as it corresponds to an exotic atom where bosonic kaons with a half-integer charge and spin are in the bound-state orbits.

A new feature in all SU(3) extensions of the Skyrme model is the fact that the Wess–Zumino interaction [3, 44, 45] contributes to the energy. As this interaction is linear in its time derivative, it can distinguish positive and negative frequencies in such a way that for a positive baryon number only the states of the negative strangeness are bound. In the SU(2) Skyrme model, the Wess–Zumino interaction is (for the same reason) essential in that it, causes the state vector of the system to change its sign under a spatial rotation by $360°$ if the number of colours is odd [3]. However there is no contribution to the energy in this case.

The extension of the model to the SU(N) group [46] represents the common structure of the Skyrme Lagrangian. Most of the studies involving the Skyrme model have concentrated on the SU(2) version of the model and its embeddings into the SU(N). However, considering SU(N) for $N \geq 3$, one has to bear in mind that the Skyrme model is not unique. In fact, there are two possible versions of the fourth-order Skyrme term. Another way is to be studied model based on the alternative form of the fourth-order Skyrme term [47].

Very little attention has been paid to field configurations describing many skyrmions in the SU(N) models which were not embeddings of the SU(2) skyrmions. Although some work has been done earlier [4, 48] the real progress has only been made since Houghton, Manton and Sutcliffe had produced their harmonic map antsatz [49]. This ansatz, when generalized to the SU(N) models [50], has lead to the construction of whole families of solutions of the SU(N) Skyrme models having spherically symmetric energy densities [51]. Moreover, it also presents field configurations, which although are not solutions of the equations, are close to them – thus providing us with good approximates to other solutions [50].

2. Nonlinear sigma model

The low energy properties of QCD mostly relevant to nuclear physics are dominated by the light quarks u, d and s. The masses of the u- and d-quarks (~ 10 MeV) are small compared to the QCD cutoff (~ 200 MeV). If these masses are neglected, the QCD Lagrangian becomes invariant under $\mathrm{SU(2)_L} \otimes \mathrm{SU(2)_R}$ chiral transformations. The absence of the parity doublets in the physical spectrum suggests that this symmetry is spontaneously broken to the $\mathrm{SU(2)_V}$ vector symmetry via the Nambu-Goldstone mechanism, with the appearance of 3 massless pseudoscalar excitations: $\pi^0, \pi^{\pm}$. In other words the QCD ground state carries an axial charge. As a result pions can decay into the vacuum.

In the absence of the quantitative understanding of non-perturbative QCD an alternative is provided at low energy by effective chiral descriptions of which the nonlinear σ-model constitutes the starting point. While QCD is undoubtedly a fundamental theory of hadrons, the chiral field models are specifically designed approximations to hadron dynamics, suited for low energy treatment.

The essence of chiral symmetry and the relative success of current algebra lies in the fact that the vacuum state in QCD spontaneously breaks the chiral symmetry. If we denote a pion state of momentum p by $|\pi^i(p)\rangle$ and the axial-vector current by $A^i_\mu(x)$ then

$$\langle 0|\, A^i_\mu(x)\, \big|\pi^j(p)\big\rangle = \mathrm{i} f_\pi p_\mu \mathrm{e}^{\mathrm{i}px}\delta^{ij}, \tag{I.2.1}$$

where $f_\pi = 130.7$ MeV is the observed pion decay constant and $\mathrm{i} = \sqrt{-1}$.

Due to the non-perturbative character of QCD in the long wavelength approximation very little is known about the pion decay constant f_π and the nucleon axial form factor g_A from first principles. There is no doubt that ultimately, lattice gauge calculations will provide a quantitative understanding of these low energy parameters. Meanwhile, the large N_c limit as advocated by 't Hooft [52] and Witten [53] is suggestive of an effective mesonic description involving the dominant chiral degrees of freedom $\pi^0, \pi^\pm$, as a substitute to QCD at low energy. Although $N_\mathrm{c} = 3$ and not infinity, one hopes that this approach will provide the relevant starting lines for discussing low energy phenomenology. In this spirit the nonlinear σ-model provides a pertinent script for chiral symmetry breaking, consistent with soft-pion threshold theorems. If we denote the scalar meson field by $\sigma(x)$ and the pseudoscalar pion field by $\boldsymbol{\pi}$, then the resulting dynamics, described by

$$\mathcal{L}_\sigma = \frac{1}{2}\left(\partial_\mu \sigma\right)^2 + \frac{1}{2}\left(\partial_\mu \boldsymbol{\pi}\right)^2, \qquad \sigma^2 + \boldsymbol{\pi}^2 = f_\pi^2, \tag{I.2.2}$$

is manifestly chiral invariant since $\left(\begin{smallmatrix}\sigma\\ \pi\end{smallmatrix}\right)$ corresponds to the $(1,0)$ representation of $\mathrm{SU(2)_L} \otimes \mathrm{SU(2)_R} \sim \mathrm{SO(4)}$.

In the trivial vacuum the nonlinear condition (I.2.2) translates into $\langle 0|\,\sigma\,|0\rangle = f_\pi$, which is the expected limit if one uses the linear σ-model with an infinitely heavy scalar particle ($m_\sigma \to \infty$). To account for the small but non vanishing mass of the pion field in nature, one adds an explicit chiral breaking term in the form $\mathcal{L}_\mathrm{Cb} = -c\sigma$. In the trivial vacuum, the pion can be understood as fluctuations of the σ-field along the valley of the tilted Mexican hat,

$$\mathcal{L}_\mathrm{Cb} = -c\sigma = -c\sqrt{f_\pi^2 - \pi^2} = -cf_\pi + \frac{c}{2}\frac{\pi^2}{f_\pi} + \mathcal{O}\left(\frac{1}{f_\pi^3}\right). \tag{I.2.3}$$

Already at this stage we can see that the nonlinear σ model satisfies the basic low energy requirements solely on the basis of chiral symmetry. It embodies an underlying topological structure that yields non-perturbative field configurations reminiscent of classical baryons.

To grasp the geometrical intricacies of the nonlinear σ-model, it is instructive to recast it in the Sugawara form [1,2]. For that, lets define the unitary 2×2 quaternion field $U(\mathbf{x})$

$$U(\mathbf{x}) = \frac{1}{f_\pi}\left(\sigma + \mathrm{i}\boldsymbol{\tau}\cdot\boldsymbol{\pi}\right), \tag{I.2.4}$$

that transforms as the $\left(\frac{1}{2},\frac{1}{2}\right)$ representation of $\mathrm{SU(2)_L} \times \mathrm{SU(2)_R}$,

$$U(\mathbf{x}) \to \exp(\mathrm{i}Q_\mathrm{L})U\exp(-\mathrm{i}Q_\mathrm{R}); \qquad Q_\mathrm{L} \in \mathrm{SU(2)_L}; \quad Q_\mathrm{R} \in \mathrm{SU(2)_R}. \tag{I.2.5}$$

In the quark picture the analogue of U^{ij} is the complex 2×2 matrix $q^{-i}[(1-\gamma_5)/2]q^j$ corresponding to pseudoscalar mesons [54]. $U(\mathbf{x}) \in$ SU(2) whose group manifold is isomorphic to S^3. The left and right currents on S^3 are defined to be

$$R_\mu = \partial_\mu U U^\dagger \to \exp(\mathrm{i}Q_\mathrm{L}) R_\mu \exp(\mathrm{i}Q_\mathrm{L}), \tag{I.2.6a}$$

$$L_\mu = U^\dagger \partial_\mu U \to \exp(\mathrm{i}Q_\mathrm{R}) L_\mu \exp(-\mathrm{i}Q_\mathrm{R}), \tag{I.2.6b}$$

which show that R_μ (L_μ) is invariant under right (left) chiral transformations. Since $\det U = 1$, it follows that

$$\partial_\mu \det U = \partial_\mu \exp \mathrm{Tr}\big(\ln U\big) = \mathrm{Tr}\big(L_\mu\big) = \mathrm{Tr}\big(R_\mu\big) = 0. \tag{I.2.7}$$

It is instructive to note that for a weak pion field L_μ and R_μ reduce to

$$L_\mu \sim -R_\mu \sim \frac{\mathrm{i}}{f_\pi} \boldsymbol{\tau} \cdot \partial_\mu \boldsymbol{\pi}. \tag{I.2.8}$$

At any fixed time, the 2×2 field $U(\mathbf{x})$ defines a map from the three dimensional space $\mathbb{R}^3$ onto the group manifold S^3, with the natural boundary condition that $U(\mathbf{x})$ goes to the trivial vacuum $\langle 0|\,\sigma\,|0\rangle = f_\pi$ at asymptotically large distances. This ensures that the energy of the corresponding field configuration is finite and

$$U(|\mathbf{x}| \to \infty) = \mathbb{1} \tag{I.2.9}$$

implies that $\mathbb{R}^3$ is compactified to S^3 as all points at infinity in $\mathbb{R}^3$ are mapped into one fixed point in S^3. The set of static maps subject to (I.2.9),

$$U(\mathbf{x}) : \mathrm{S}^3 \to \mathrm{S}^3, \tag{I.2.10}$$

is known to be non trivial. In other words, at a given time, it is possible to split the set of all maps into homotopically distinct classes not continuously deformable into each other. These classes are called the homotopy or Chern-Pontryagin classes. In this case, they constitute the third homotopy group $\pi_3(\mathrm{S}^3) \sim \mathbb{Z}$, where $\mathbb{Z}$ is the additive group of integers. It is these integers that are referred to as winding numbers of the mapping $U(\mathbf{x})$. Since a continuous evolution in time can be understood as a homotopy transformation, the corresponding winding numbers are conserved by definition independently of the details of the underlying dynamics.

In order to construct an explicit form of the topological charge $\mathcal{B}^0$ in the nonlinear σ-model:

$$\mathcal{B}^0 : \pi_3(\mathrm{S}^3) \to \mathbb{Z}, \tag{I.2.11}$$

it is convenient to use the vector representation $(1,0)$ of $\mathrm{SU(2)_L} \otimes \mathrm{SU(2)_R}$,

$$\phi^0 = \frac{\sigma}{f_\pi}, \qquad \phi^i = \frac{\pi^i}{f_\pi}. \tag{I.2.12}$$

An elementary surface element in the group manifold is characterized by

$$\mathrm{d}^3\Sigma = \epsilon^{ijkl} \phi^i \partial_1 \phi^j \partial_2 \phi^k \partial_3 \phi^l \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3, \tag{I.2.13}$$

where (x^1, x^2, x^3) are the corresponding coordinates on $\mathbb{R}^3$ obtained by stereographic projection from S^3. Equation (I.2.13) is just the Jacobian associated to the affine transformation $\mathrm{S}^3 \to \mathrm{S}^3$. Hence, the normalized topological density reads

$$\mathcal{B}^0 = \frac{1}{3! A_3}\, \epsilon^{ijkl} \epsilon^{\nu\alpha\beta} \phi^i \partial_\nu \phi^j \partial_\alpha \phi^k \partial_\beta \phi^l, \tag{I.2.14}$$

where $A_3 = 2\pi^2$ is the surface of S^3 in $\mathbb{R}^4$. To rewrite the equation (I.2.14) in terms of (I.2.6), notice that for a weak pion field, i.e. $\phi^0 \sim 1/f_\pi$ and $\phi^i \sim \pi^i/f_\pi$. We have

$$\begin{aligned}\mathcal{B}^0 &= \frac{(-\mathrm{i})^3}{24\pi^2}\,\epsilon^{\nu\alpha\beta}\,\mathrm{Tr}\left(\partial_\nu\left(\frac{\mathrm{i}\boldsymbol{\tau}\cdot\boldsymbol{\pi}}{f_\pi}\right)\partial_\alpha\left(\frac{\mathrm{i}\boldsymbol{\tau}\cdot\boldsymbol{\pi}}{f_\pi}\right)\partial_\beta\left(\frac{\mathrm{i}\boldsymbol{\tau}\cdot\boldsymbol{\pi}}{f_\pi}\right)\right) + \mathcal{O}\left(\frac{1}{f_\pi^4}\right)\\ &= -\frac{\mathrm{i}}{24\pi^2}\,\epsilon^{0\nu\alpha\beta}\,\mathrm{Tr}\Big(R_\nu R_\alpha R_\beta\Big). \end{aligned} \tag{I.2.15}$$

This equation follows from the chiral invariance. Notice that $\mathcal{B}$ does not vanish if and only if the 3 pion degrees of freedom, $\pi^0, \pi^\pm$, are excited. It is rather obvious from the above construction that

$$B = \int_{\mathbb{R}^3} \mathcal{B}^0 \mathrm{d}x = \text{winding number}. \tag{I.2.16}$$

The covariant current associated to the topological charge (I.2.15) is given by

$$\mathcal{B}^\mu = -\frac{\mathrm{i}}{24\pi^2}\epsilon^{\mu\nu\beta\gamma}\,\mathrm{Tr}\Big(R_\nu R_\beta R_\gamma\Big), \tag{I.2.17}$$

and is conserved almost everywhere in $\mathbb{R}^3$ independently of the equations of motion, i.e. $\partial^\mu \mathcal{B}_\mu = 0$.

The topological charge (I.2.15) bears some similarities with the monopole and instanton charges. It follows from the underlying Riemannian structure of the group manifold and is conserved regardless of the equations of motion. Its form is obtained by an explicit construction of the isomorphism: $\pi_3(\mathrm{S}^3) \sim \mathbb{Z}$. The topological charge has no dual charge, and can be localized arbitrarily in space since (I.2.15) is not a total divergence.

3. The Skyrme model

So far the nonlinear σ-model has provided a rather economical framework for discussing low energy phenomenology. Its finite energy configuration space exhibits a nontrivial topological structure due to the underlying Riemannian geometry of the coset space. As a consequence, there exist static and finite field configurations other than the trivial vacuum, that are characterized by conserved topological charges. Unfortunately, these configurations are not energetically stable in 3 dimensional space according to Derrick's theorem [55, 56]. Indeed, if $U(\mathbf{x})$ is a static field configuration solution to the Euler-Lagrange equations associated to

$$\mathcal{L}_\sigma = -\frac{f_\pi^2}{4}\,\mathrm{Tr}\Big(R_\mu R^\mu\Big), \tag{I.3.18}$$

which is the Sugawara's form of (I.2.2), then

$$E = \int \mathrm{d}^D x \frac{f_\pi^2}{4}\,\mathrm{Tr}\Big(\partial^i U^\dagger \partial^i U\Big). \tag{I.3.19}$$

A simple rescaling of $U(\mathbf{x})$ in space, i.e. $U(\mathbf{x}) \to U(\lambda\mathbf{x})$, yields

$$E_\lambda = \lambda^{2-D} E. \tag{I.3.20}$$

Equation (I.3.20) clearly shows that for $D = 3$, the energetically favorable configurations have zero energy. In other words the finite energy solutions of the nonlinear σ-model are unstable against scale transformations.

To avoid this collapse, Skyrme added to (I.3.18) a quartic term in the currents R_μ (the Skyrme term),

$$\mathcal{L}_{\text{Sk}} = -\frac{f_\pi^2}{4}\,\text{Tr}\Big(R_\mu R^\mu\Big) + \frac{1}{32e^2}\,\text{Tr}\Big([R_\mu, R_\nu][R^\mu, R^\nu]\Big). \tag{I.3.21}$$

Here e is a dimensionless parameter that characterizes the size of the finite energy configurations. The values of parameters f_π and e are fixed by comparison with the experimental data.

The Euler-Lagrange equation which follows from (I.3.21) is the Skyrme field equation

$$\partial_\mu\left(R^\mu + \frac{1}{8f_\pi^2 e^2}\big[R^\nu, [R_\nu, R^\mu]\big]\right) = 0, \tag{I.3.22}$$

which is a nonlinear wave equation for $U(\mathbf{x})$. The Bogomolny construction to simplify (I.3.22) via saturation mechanism to obtain analytical solutions (a well known procedure for instantons) cannot be used in this case. A numerical treatment of (I.3.22) is required.

Since the Skyrme term scales like r^{-4} in the three dimensional space it will prevent the Skyrme solitons from collapsing to zero size. Indeed, a rescaling in space of any finite energy solution of (I.3.21) translates to a rescaling in the ground state energy in the form

$$E_\lambda = \lambda^{2-D} E_{(2)} + \lambda^{4-D} E_{(4)}. \tag{I.3.23}$$

It is straightforward to show that E_λ exhibits a true minimum for $D \geq 3$, i.e.,

$$\frac{\mathrm{d}E_\lambda}{\mathrm{d}\lambda} = 0 \to \frac{E_{(2)}}{E_{(4)}} = -\frac{D-4}{D-2}\,, \tag{I.3.24a}$$

$$\frac{\mathrm{d}^2 E_\lambda}{\mathrm{d}\lambda^2} > 0 \to 2(D-2)E_{(2)} > 0\,. \tag{I.3.24b}$$

In particular for $D = 3$ the equation (I.3.24a) shows that $E_{(2)} = E_{(4)}$, as expected from the virial theorem. Due to the underlying geometry, the energy is bounded from below by the topological charge. Indeed, for a static configuration

$$E = \int \mathrm{d}^3x \left(-\frac{f_\pi^2}{4}\,\text{Tr}\Big(R_i^2\Big) - \frac{1}{32e^2}\,\text{Tr}\Big([R_i, R_j]^2\Big)\right). \tag{I.3.25}$$

A lower bound to (I.3.25) can be obtained using the following inequality,

$$\int \mathrm{d}^3x\,\text{Tr}\left(\frac{f_\pi^2}{2}R_i^2 + \frac{1}{8e^2}\epsilon_{ijk}[R_j, R_k]\right)^2 \leq 0. \tag{I.3.26}$$

where this facts are used that the R_i's are anti-Hermitian matrices and that for any anti-Hermitian matrix G, $\text{Tr}\big(G^2\big) \leq 0$. In terms of the topological charge B, this becomes

$$E \geq \frac{6\pi^2 f_\pi}{e}\,|B|\,. \tag{I.3.27}$$

Equation (I.3.27) is sometimes referred to as the Bogomolny bound [57]. Since there are no self dual chiral fields, the energy is strictly larger than the estimate (I.3.27). For skyrmions the Bogomolny bound cannot be saturated. Equation (I.3.27) illustrates in a striking way the mechanism by which geometry induces local stability at the classical level.

The Skyrme term can be understood as a higher-order correction to the nonlinear σ-model when cast in the general framework of an effective chiral description as advocated by Weinberg [58]. While the quadratic term (I.3.18) in the nonlinear σ-model is only one chirally invariant term of the second order, the fourth order term is not unique to order chirally invariance. Indeed, under the general assumptions of locality, the Lorentz invariance, the chiral symmetry, the parity and the G-parity there are 3 independent terms to order chirally invariance, i.e.

$$\mathcal{L}_{(4)} = a\,\mathrm{Tr}\Big(R_\mu R^\mu R_\nu R^\nu\Big) + b\,\mathrm{Tr}\Big(R_\mu R_\nu R^\mu R^\nu\Big) + c\,\mathrm{Tr}\Big(\partial_\mu R_\nu \partial^\mu R^\nu\Big), \tag{I.3.28}$$

where a, b, c are some constants. Other combinations can be eliminated by the Maurer-Cartan equation for the right current on S^3

$$\partial_\mu R_\nu - \partial_\nu R_\mu + [R_\mu, R_\nu] = 0, \tag{I.3.29}$$

which follows trivially from the definitions (I.2.6). To this order, the combination:

$$\mathcal{L}_{(4)} = \mathrm{Tr}\Big(R_\mu R^\mu R_\nu R^\nu\Big) - \mathrm{Tr}\Big(R_\mu R_\nu R^\mu R^\nu\Big) = -\frac{1}{2}\,\mathrm{Tr}\Big([R_\mu, R_\nu][R^\mu, R^\nu]\Big), \tag{I.3.30}$$

is the unique term with four derivatives that leads to a Hamiltonian having the second order in time derivatives. (I.3.30) is exactly the term suggested by T. H. R. Skyrme [59]. This is important because there are problems with the stability of the classical solution, once higher order terms in the time derivatives are included.

After the baryon number and energy, the most significant characteristic of the static solution of the Skyrme equation is its asymptotic field, which satisfies the linearized form of the equation. To the leading order, each of the three components of the pion field $\boldsymbol{\pi}$ obey Laplace's equation, and σ can be taken to be unity. More precisely, $\boldsymbol{\pi}$ has a multipole expansion, in which each term is an inverse power of $r = |\mathbf{x}|$, say $r^{-(l+1)}$, times a triplet of angular functions. The leading term, with the smallest l, obeys Laplace's equation, whereas subleading terms may not, because of the nonlinear aspect of the Skyrme equation. For the leading term, therefore, the angular functions are a triplet of the linear combinations of the spherical harmonics $Y_{l,m}(\theta, \varphi)$, with m taking integer values in the range $-l \leq m \leq l$. These spherical harmonics can also be expressed in the Cartesian coordinates, which often gives more convenient and elegant formulae for the asymptotic fields.

One of the few precise results concerning the Skyrme equation (I.3.22) is that this multipole expansion can not lead with a monopole term, having $l = 0$. The leading term is a dipole or higher multipole [60].

Recently, it has been rigorously proven [61] that for any non-vacuum soliton of the Skyrme equation, the multipole expansion is non-trivial. In other words, the pion field does not vanish to all orders in l, and the leading term is a multipole satisfying the Laplace equation.

The skyrmion

In his pioneering work Skyrme [1] believed that the field configurations of the winding number one ($B = 1$) were fermions. He conjectured that the topological current (I.2.17) should be identified with the baryon current, suggesting that skyrmions were classical baryons.

The classical Skyrme model is formulated in the arbitrary irreducible SU(2) representation j [11]. The Euler angles $\alpha(\mathbf{x}, t)$ become the functions of the space-time point $(\mathbf{x}, t)$ and form the dynamical variables of the theory. Model is formulated in terms of a unitary field

$$U(\mathbf{x}, t) = D^j\big(\alpha(\mathbf{x}, t)\big), \tag{I.3.31}$$

where symbol D^j denotes the SU(2) Wigner matrices.

By using definitions from Appendix A allow us to express the Lagrangian density (I.3.21) in terms of the Euler angles

$$\begin{aligned}\mathcal{L}_{\text{Sk}} = & \frac{1}{3}j(j+1)(2j+1)\Bigg(\frac{f_\pi^2}{4}\Big(\partial_\mu\alpha^i\partial^\mu\alpha^i + 2\cos\alpha^2\,\partial_\mu\alpha^1\,\partial^\mu\alpha^3\Big) \\ & - \frac{1}{16e^2}\Big(\partial_\mu\alpha^2\partial^\mu\alpha^2(\partial_\nu\alpha^1\partial^\nu\alpha^1 + \partial_\nu\alpha^3\partial^\nu\alpha^3) - (\partial_\mu\alpha^1\partial^\mu\alpha^2)^2 \\ & \qquad - (\partial_\mu\alpha^2\partial^\mu\alpha^3)^2 + \sin^2\alpha^2(\partial_\mu\alpha^1\partial^\mu\alpha^1\,\partial_\nu\alpha^3\partial^\nu\alpha^3 - (\partial_\mu\alpha^1\partial^\mu\alpha^3)^2) \\ & \qquad + 2\cos\alpha^2(\partial_\mu\alpha^2\partial^\mu\alpha^2\,\partial_\nu\alpha^1\partial^\nu\alpha^3 - \partial_\mu\alpha^1\partial^\mu\alpha^2\partial_\nu\alpha^2\partial^\nu\alpha^3)\Big)\Bigg).\end{aligned} \tag{I.3.32}$$

The only dependence on the dimension of the representation is in the overall factor $j(j+1)(2j+1)$ as it could be expected from the inner product of two SU(2) generators. This implies that the equation of motion for the dynamical variable α is independent of the dimension of the representation.

In terms of the Euler angles α the baryon current density takes the form

$$\begin{aligned}\mathcal{B}^\mu &= \frac{1}{24N\pi^2}\epsilon^{\mu\nu\beta\gamma}\,\mathrm{Tr}\Big(R_\nu R_\beta R_\gamma\Big) \\ &= -\frac{1}{24\cdot 6N\pi^2}j(j+1)(2j+1)\sin\alpha^2\,\epsilon^{\mu\nu\beta\gamma}\,\epsilon_{ikl}\,\partial_\nu\alpha^i\partial_\beta\alpha^k\partial_\gamma\alpha^l.\end{aligned} \tag{I.3.33}$$

The normalization factor N depends on the dimension of the representation and has the value 1 in the fundamental ($j = 1/2$) representation. As the dimensionality of the representation appears in this expression in the same overall factor as in the Lagrangian density (I.3.32) it follows that all calculated dynamical observables will be independent of the dimension of the representation at the classical level.

The equation of motion (I.3.22), which follows from the variation of the Lagrangian (I.3.21), are highly nonlinear and so far can only be handled under the assumption of maximal symmetry as suggested by Skyrme's static hedgehog ansatz [2]

$$\begin{aligned}U(x) &= \exp\left(\mathrm{i}\boldsymbol{\tau}\cdot\hat{\mathbf{r}}F(r)\right) \\ &= \cos F(r) + \mathrm{i}\boldsymbol{\tau}\cdot\hat{\mathbf{r}}\sin F(r).\end{aligned} \tag{I.3.34}$$

The name hedgehog derives from the fact that the pion fields of this configuration point radially outward from the origin at all points in space, so $\hat{\boldsymbol{\pi}} = \hat{\mathbf{r}}$ where $\hat{\mathbf{r}}$ denotes the unit spatial vector. $F(r)$ is the real radial profile function with the boundary conditions $F(0) = \pi$ and $F(\infty) = 0$. The latter condition ensures that $U(\infty) = \mathbb{1}$, while the former guarantees that $U(0)$ is well defined and that $B = 1$. The object described by (I.3.34) has a peculiar geometric structure which is shown in Fig. 1.

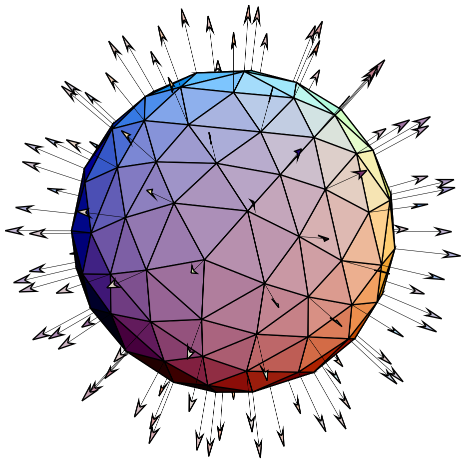

Figure 1: The hedgehog configuration [62].

The Equation (I.3.34) follows from the fact that in a given topological sector, the maximal compact symmetry group of the configuration space is

$$\mathrm{diag}\left(\mathrm{SU}(2)_{\mathrm{L}} \otimes \mathrm{SU}(2)_{\mathrm{R}}\right) \sim \mathrm{diag}\left(\mathrm{SO}(2)_I \otimes \mathrm{SO}(3)_J\right), \tag{I.3.35}$$

where $\mathrm{SO}(3)_J$ and $\mathrm{SO}(3)_I$ refer to the orthogonal group of rotations in space and isospace respectively. In the ansatz (I.3.34), the isospin (I) and the angular momentum (J) are correlated in a way that neither of them is good quantum number, but their sum is

$$\boldsymbol{K} = \boldsymbol{J} + \boldsymbol{I} \equiv (\boldsymbol{L} + \boldsymbol{S}) + \boldsymbol{I}. \tag{I.3.36}$$

$U(\mathbf{x})$ is left invariant under rotations in K-space, i.e.

$$\begin{aligned}\left[\boldsymbol{K}, U(\mathbf{x})\right] &= \mathrm{i}\sin F\left(\left[\left(\mathbf{r}\times\frac{\nabla}{\mathrm{i}}\right), \boldsymbol{\tau}\cdot\hat{\mathbf{r}}\right] + \left[\boldsymbol{\tau}/2, \boldsymbol{\tau}\cdot\hat{\mathbf{r}}\right]\right)\\ &= \mathrm{i}\sin F\left(-\mathrm{i}\left(\boldsymbol{\tau}\times\hat{\mathbf{r}}\right) - \mathrm{i}\left(\hat{\mathbf{r}}\times\boldsymbol{\tau}\right)\right) \equiv 0,\end{aligned} \tag{I.3.37}$$

suggesting that hedgehog skyrmions are scalar in K-space $(K = 0)$. Since parity is defined through

$$\hat{\boldsymbol{\pi}}_{\mathrm{op}} U(\mathbf{x}, t)\hat{\boldsymbol{\pi}}_{\mathrm{op}}^{-1} = U^{\dagger}(-\mathbf{x}, t), \tag{I.3.38}$$

one concludes that the ansatz (I.3.34) is parity invariant. Here $\hat{\boldsymbol{\pi}}_{\mathrm{op}}$ is the parity operator. Consequently, hedgehog skyrmions carry $\boldsymbol{K}^{\pi} = 0^{+}$ and can be viewed as a mixture of states with $I = J$ and the positive parity.

The substitution of the hedgehog ansatz (I.3.34) into the Lagrangian density (I.3.21) leads to the following form:

$$\begin{aligned}\mathcal{L}_{\mathrm{Sk}} = -\frac{4}{3}j(j+1)(2j+1)\Bigg(&\frac{f_\pi^2}{4}\left(F'^2 + \frac{2}{r^2}\sin^2 F\right)\\ &+ \frac{1}{4e^2}\frac{\sin^2 F}{r^2}\left(2F'^2 + \frac{\sin^2 F}{r^2}\right)\Bigg).\end{aligned} \tag{I.3.39}$$

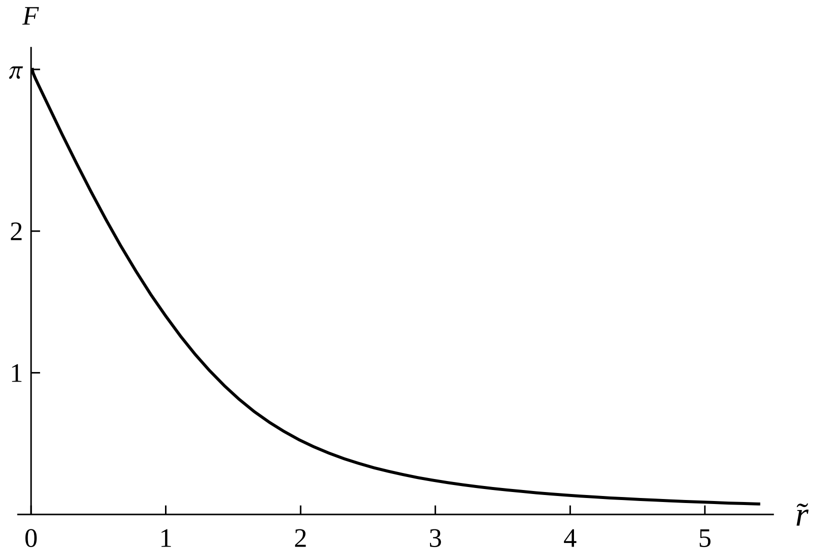

Figure 2: A classical solution of the profile function $F(\tilde{r})$ for the $B = 1$ skyrmion.

For the case of $j = 1/2$ this reduces to the result of [5]. The corresponding mass density is obtained by reverting the sign of $\mathcal{L}_{\text{Sk}}$, as the hedgehog ansatz is a static solution.

The requirement that the soliton mass is stationary yields the following nonlinear ordinary differential equation:

$$f_\pi^2\left(F'' + \frac{2}{r}F' - \frac{\sin 2F}{r^2}\right) - \frac{1}{e^2}\left(\frac{1}{r^4}\sin^2 F\,\sin 2F - \frac{1}{r^2}\left(F'^2\sin 2F + 2F''\sin^2 F\right)\right) = 0. \quad \text{(I.3.40)}$$

It is independent of the dimension of the representation. Note that the differential equation is nonsingular only if $F(0) = n\pi,\ n \in \mathbb{Z}$.

Indeed, the baryon density for the hedgehog configuration becomes

$$\mathcal{B}^0 = -\frac{1}{3N\pi^2}j(j+1)(2j+1)\frac{\sin^2 F}{r^2}F', \quad \text{(I.3.41)}$$

leading to the baryon number of the form

$$B = \int \mathrm{d}^3 r \mathcal{B}^0 = \frac{2}{3N\pi^2}j(j+1)(2j+1)\big(F(0) - \frac{1}{2}\sin 2F(0)\big). \quad \text{(I.3.42)}$$

A combination of the requirement that $F(0)$ is an integer multiple of π and the requirement that the lowest nonvanishing baryon number is 1, gives a general expression for the normalization factor N as

$$N = \frac{2}{3}j(j+1)(2j+1). \quad \text{(I.3.43)}$$

The equation of motion for the profile function in the form of (I.3.40) depends on the parameter f_π and e values. It is convenient to introduce a dimensionless variable $\tilde{r} = ef_\pi r$ in which

(I.3.40) takes the form

$$F''(\tilde{r})\left(1+\frac{2\sin^2 F(\tilde{r})}{\tilde{r}^2}\right)+F'^2(\tilde{r})\frac{\sin 2F(\tilde{r})}{\tilde{r}^2}+\frac{2}{\tilde{r}}F'(\tilde{r}) \\ -\frac{\sin 2F(\tilde{r})}{\tilde{r}^2}-\frac{\sin 2F(\tilde{r})\sin^2 F(\tilde{r})}{\tilde{r}^4}=0. \quad \text{(I.3.44)}$$

The solution of this equation, satisfying the boundary conditions, can not be obtained in closed form but it is a simple task to compute it numerically. A numerical investigation of this equation leads to the classical profile function solution $F(\tilde{r})$ shown in Fig. 2, when boundary conditions are $F(0)=\pi$ and $F(\infty)=0$, and the baryon number $B=1$.

Skyrme model involving higher order derivatives

The Skyrme model can be generalized by adding terms involving higher order derivatives in the Lagrangian (I.3.21) [23, 63, 64]. By doing this, one introduces extra parameters that can be tuned to increase the quality of the Skyrme model as an effective low energy limit of QCD (via $\frac{1}{N_c}$ expansion [53], chiral bozonization [65], etc.) and that all parameters could be determined from it. On the other hand, it serves no practical purpose if we need to fit a large numbers of parameters fixed by experiment measurements since the model would loose much of all its predictive power.

A large-N_c analysis suggests that the bosonization of QCD would most likely involve an infinite number of mesons. And if this is the case, then taking the appropriate decoupling limits (or large-mass limits) for higher spin mesons leads to an all-orders Lagrangian for pions. From the point of view of the QCD perturbation theory, one can also expect such terms in hadron interactions since they are connected to higher twist effects. One example of higher order terms is the piece proposed by Jackson *et al.* [23]:

$$\mathcal{L}_6 = c_6\, \epsilon^{\mu\nu_1\nu_2\nu_3}\, \epsilon_{\mu\lambda_1\lambda_2\lambda_3}\, \mathrm{Tr}\Big(R_{\nu_1}R_{\nu_2}R_{\nu_3}R^{\lambda_1}R^{\lambda_2}R^{\lambda_3}\Big). \quad \text{(I.3.45)}$$

As in the case of the quartic term one can construct an alternative sixth order term, which is equivalent to (I.3.45) in the case of the fundamental representation

$$\mathcal{L}_6 = c_6'\, \mathrm{Tr}\Big([R_\mu, R^\nu][R_\nu, R^\lambda][R_\lambda, R^\mu]\Big). \quad \text{(I.3.46)}$$

The unknown coefficients c_6 and c_6' denote the strength of those terms and are free parameters of the model. This sixth-order term preserves the Lorentz invariance and the SU(N) symmetry of the model and leads to an equation of motion that does not involve derivatives of the order higher than two.

In terms of the Euler angles α this Lagrangian density takes the form [11]

$$\mathcal{L}_6 = -c_6'\frac{j(j+1)(2j+1)}{6}\epsilon_{i_1i_2i_5}\,\epsilon_{i_3i_4i_6}\,\sin^2\alpha^2 \\ \times\,\partial_\mu\alpha^{i_1}\,\partial^\nu\alpha^{i_2}\,\partial_\nu\alpha^{i_3}\partial^\lambda\alpha^{i_4}\,\partial_\lambda\alpha^{i_5}\,\partial^\mu\alpha^{i_6}. \quad \text{(I.3.47)}$$

This result reveals that the dependence on the dimension of the representation of this term is contained in the same overall factor $j(j+1)(2j+1)$ as in the Skyrme model Lagrangian (I.3.21).

Hence the addition of the term $\mathcal{L}_6$ maintains the simple overall dimension dependent factor of the original Skyrme model.

Studies of the Skyrme model by adding a sixth-order term (I.3.46) to the Lagrangian have shown that the multi-skyrmion solutions of the extended model have the same symmetry as the pure Skyrme model [66]. Also that the addition of the sixth-order term makes the multi-skyrmion solution more bound than in the pure Skyrme model and that it also reduces the solution radius. If used in the extended model, the harmonic map ansatz for the multi-skyrmions works as well or even better, than for the pure Skyrme model.

On the other hand, several attempts were made to incorporate vector mesons in the Skyrme picture. These procedures are characterized by the addition of a piece to the Lagrangian describing free vector mesons, typically of the form of an SU(2) gauge Lagrangian $\mathrm{Tr}\,(F_{\mu\nu}F^{\mu\nu})$ and the substitution of the derivative by a covariant derivative to account for scalar-vector interactions. In the large-mass limit of the vector mesons, they decouple and an effective self-interaction for scalar mesons is induced as $F_{\mu\nu} \to f_{\mu\nu} \equiv [R_\mu, R_\nu]$.

Following this approach Marleau studied the model where a large number of higher order terms were included in the Lagrangian [63, 64]. He has shown that an infinite class of alternate stabilizing terms for the Lagrangian density exists. They involve all orders in the derivatives of the pion field, but their energy densities are only second order in the derivative of the profile function $F(r)$. Summing to all orders, the mass of the static solution takes a general form

$$M_{\mathrm{cl}} = 8\pi \int r^2 \mathrm{d}r \sum_{m=1}^{\infty} \mathrm{c}_m \left(\frac{\sin^2 F}{r^2}\right)\left(3\frac{\sin^2 F}{r^2} + m\left(F'^2 - \frac{\sin^2 F}{r^2}\right)\right), \tag{I.3.48}$$

where c_m are free parameters of the model and $\mathrm{c}_m = 0$ for any odd $m \geq 5$. The differential equation for the profile function then reads

$$\sum_{m=1}^{\infty} m\,\mathrm{c}_m \left(\frac{\sin^2 F}{r^2}\right)^{m-1}\Bigg(F'' + 2(2-m)\frac{F'}{r} + (m-1)F'^2\frac{\cos F}{\sin F} - (3-m)\frac{\sin F\cos F}{r^2}\Bigg) = 0. \tag{I.3.49}$$

4. The rational map approximation

It has been found that many solutions of the Skyrme equation, and particularly those of low energy, look like monopoles, with the baryon number B being identified with the monopole number N. Of course, it is not expected that an exact correspondence exists, since the Yang-Mills-Higgs and Skyrme models have a number of very different properties and the fields are not really the same, but the energy density has equivalent symmetries and approximately the same spatial distribution. As yet, there is no known direct transformation between the fields of a monopole and those of a skyrmion, but there is an indirect transformation via rational maps between the Riemann spheres.

A rational map is a holomorphic function from $\mathrm{S}^2 \mapsto \mathrm{S}^2$. If we treat each S^2 as a Riemann sphere, the first having a coordinate z, a rational map of degree N is a function $R : \mathrm{S}^2 \mapsto \mathrm{S}^2$ where

$$R(z) = \frac{p(z)}{q(z)} \tag{I.4.1}$$

and p and q are polynomials of degree at most N. Either p or q must have its degree precisely N, and p and q must have no common roots, otherwise factors can be cancelled between them. q can be a non-zero constant, in this case R is just a polynomial. For finite z, $R(z)$ may have any complex value, including infinity. The value is infinity where q vanishes. $R(\infty)$ is the limit of $p(z)/q(z)$ as $z \mapsto \infty$ and can be either finite or infinite.

Rational maps are maps from $\mathrm{S}^2 \mapsto \mathrm{S}^2$, whereas skyrmions are maps from $\mathbb{R}^3 \mapsto \mathrm{S}^3$. The main idea behind the rational map ansatz, introduced in [49], is to identify the domain S^2 of the rational map with the concentric spheres in $\mathbb{R}^3$, and the target S^2 with the spheres of latitude on S^3.

It is convenient to use the Cartesian notation to present the ansatz. Recall that via a stereographic projection, the complex coordinate z on a sphere can be identified with conventional polar coordinates by $z = \tan(\theta/2)\mathrm{e}^{\mathrm{i}\varphi}$. Equivalently, the point z corresponds to the unit vector

$$\hat{\mathbf{n}}_z = \frac{1}{1+|z|^2}\Big\{2\Re(z), 2\Im(z), 1-|z|^2\Big\}. \tag{I.4.2}$$

Similarly the value of the rational map $R(z)$ is associated with the unit vector

$$\hat{\mathbf{n}}_R = \frac{1}{1+|R|^2}\Big\{2\Re(R), 2\Im(R), 1-|R|^2\Big\}. \tag{I.4.3}$$

Let us denote a point in $\mathbb{R}^3$ by its coordinates (r, z) where r is the radial distance from the origin and z specifies the direction from the origin. The ansatz for the Skyrme field depends on a rational map $R(z)$ and a profile function $F(r)$. The ansatz is

$$U(r, z) = \exp(\mathrm{i}F(r)\ \hat{\mathbf{n}}_R \cdot \boldsymbol{\tau}) \tag{I.4.4}$$

where $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$ denotes the Pauli matrices. $U(r, z)$ is well-defined at the origin, if $F(0) = k\pi$, for some integer k. The boundary value $U = \mathbb{1}$ at $r = \infty$ requires that $F(\infty) = 0$. It is straightforward to verify that the baryon number of this field is $B = Nk$, where N is the degree of R. In the remainder of this section we shall consider only the case $k = 1$, and consequently $B = N$.

An SU(2) Möbius transformation on the domain S^2 of the rational map corresponds to a spatial rotation

$$R(z) \mapsto \frac{\alpha R(z) + \beta}{-\bar{\beta}R(z) + \bar{\alpha}}, \qquad \text{where} \qquad |\alpha|^2 + |\beta|^2 = 1, \tag{I.4.5}$$

whereas an SU(2) Möbius transformation on the target S^2 corresponds to a rotation of $\hat{\mathbf{n}}_R$, and hence to an isospin rotation of the Skyrme field. Thus if a rational map $R : \mathrm{S}^2 \mapsto \mathrm{S}^2$ is symmetric (i.e. a rotation of the domain can be compensated by a rotation of the target), then the resulting Skyrme field is symmetric (i.e. a spatial rotation can be compensated by an isospin rotation).

In the case of $N = 1$, the basic map is $R(z) = z$, which is spherically symmetric, and (I.4.4) reduces to Skyrme's hedgehog field

$$U(r, \theta, \varphi) = \cos F + \mathrm{i}\sin F(\sin\theta\cos\varphi\ \tau_1 + \sin\theta\sin\varphi\ \tau_2 + \cos\theta\ \tau_3). \tag{I.4.6}$$

As in nonlinear elasticity theory, the energy density of a Skyrme field depends on the local stretching associated with the map $U : \mathbb{R}^3 \mapsto \mathrm{S}^3$. The Riemannian geometry of $\mathbb{R}^3$ (flat) and of

S^3 (a unit radius 3-sphere) are necessary to define this stretching. Consider the strain tensor at a point in $\mathbb{R}^3$

$$D_{ij} = -\frac{1}{2}\,\mathrm{Tr}\Big(R_iR_j\Big) = -\frac{1}{2}\,\mathrm{Tr}\Big((\partial_i UU^{-1})(\partial_j UU^{-1})\Big). \tag{I.4.7}$$

It is symmetric and positive semi-definite as R_i is antihermitian. Let its eigenvalues be λ_1^2, λ_2^2 and λ_3^2. The Skyrme energy can be reexpressed as

$$E = \int (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_1^2\lambda_3^2)\, \mathrm{d}^3x, \tag{I.4.8}$$

and the baryon density as $\lambda_1\lambda_2\lambda_3/2\pi^2$. For the ansatz (I.4.4), the strain in the radial direction is orthogonal to the strain in the angular directions. Moreover, because $R(z)$ is conformal, the angular strains are isotropic. If we identify λ_1^2 with the radial strain and λ_2^2 and λ_3^2 with the angular strains, we can compute that

$$\lambda_1 = -F'(r), \qquad \lambda_2 = \lambda_3 = \frac{\sin F}{r}\frac{1+|z|^2}{1+|R|^2}\left|\frac{\mathrm{d}R}{\mathrm{d}z}\right|. \tag{I.4.9}$$

Therefore the energy is

$$E = \int\Bigg(F'^2 + 2(F'^2+1)\frac{\sin^2 F}{r^2}\left(\frac{1+|z|^2}{1+|R|^2}\left|\frac{\mathrm{d}R}{\mathrm{d}z}\right|\right)^2 + \frac{\sin^4 F}{r^4}\left(\frac{1+|z|^2}{1+|R|^2}\left|\frac{\mathrm{d}R}{\mathrm{d}z}\right|\right)^4\Bigg)\frac{2\mathrm{i}\mathrm{d}z\mathrm{d}\bar{z}r^2\mathrm{d}r}{(1+|z|^2)^2}, \tag{I.4.10}$$

where $2\mathrm{i}\mathrm{d}z\mathrm{d}\bar{z}/(1+|z|^2)^2$ is equivalent to the usual area element on a 2-sphere $\sin\theta\mathrm{d}\theta\mathrm{d}\varphi$. Now the part of the integrand

$$\left(\frac{1+|z|^2}{1+|R|^2}\left|\frac{\mathrm{d}R}{\mathrm{d}z}\right|\right)^2\frac{2\mathrm{i}\mathrm{d}z\mathrm{d}\bar{z}}{(1+|z|^2)^2} \tag{I.4.11}$$

is precisely the pull-back of the area form $2\mathrm{i}\mathrm{d}R\mathrm{d}\bar{R}/(1+|R|^2)^2$ on the target sphere of the rational map R; therefore its integral is 4π times the degree N of R. So the energy simplifies to

$$E = 4\pi\int\left(r^2F'^2 + 2N(F'^2+1)\sin^2 F + \mathcal{I}\,\frac{\sin^4 F}{r^2}\right)\mathrm{d}r \tag{I.4.12}$$

where $\mathcal{I}$ denotes the integral

$$\mathcal{I} = \frac{1}{4\pi}\int\left(\frac{1+|z|^2}{1+|R|^2}\left|\frac{\mathrm{d}R}{\mathrm{d}z}\right|\right)^4\frac{2\mathrm{i}\mathrm{d}z\mathrm{d}\bar{z}}{(1+|z|^2)^2}. \tag{I.4.13}$$

$\mathcal{I}$ depends only on the rational map R. It appears that $\mathcal{I}$ is a “proper” Morse function, that is, the set of rational maps, and hence monopoles, for which $\mathcal{I}$ has any particular finite value is compact.

To minimize E for maps of a given degree N, one should first minimize $\mathcal{I}$ over all maps of degree N. Then, the profile function $F(r)$ minimizing the energy (I.4.12) may be found by solving a second order differential equation with N and $\mathcal{I}$ as parameters.

An important quantity associated with a rational map $R(z)$ is the Wronskian

$$W(z) = p'(z)q(z) - q'(z)p(z) \tag{I.4.14}$$

or more precisely, the zeros of W, which are the branch points of the map. If R is of degree N, then generically, W is a polynomial of degree $2N-2$. The zeros of W are invariant under any Möbius transformation of R, which replaces p by $\alpha p+\beta q$ and q by $\gamma q+\delta p$ and hence simply multiplies W by $(\alpha\gamma-\beta\delta)$. Occasionally, W is a polynomial of degree less than $2N-2$, but one then interprets the missing zeros as being at $z=\infty$. The symmetries of the map R are captured by the symmetries of the Wronskian W. Sometimes W has more symmetries than the rational map R.

The zeros of the Wronskian $W(z)$ of a rational map $R(z)$ give interesting information about the shape of the Skyrme field which is constructed from R using the ansatz (I.4.4). Where W is zero, the derivative $\mathrm{d}R/\mathrm{d}z$ is also zero, so the strain eigenvalues in the angular directions, λ_2 and λ_3, vanish. The baryon density, being proportional to $\lambda_1\lambda_2\lambda_3$, vanishes along the entire radial line in the direction specified by any zero of W. The energy density will also be low along such a radial line, since there will only be the contribution λ_1^2 from the radial strain eigenvalue. The Skyrme field baryon density contours will therefore look like a polyhedron with holes in the directions given by the zeros of W, and there will be $2N-2$ of such holes. This structure is seen in all the plots shown in Fig. 3. For example, the $B=7$ skyrmion having twelve holes arranged at the face centres of a dodecahedron.

A rational map, $R:\mathrm{S}^2\mapsto\mathrm{S}^2$, is invariant (or symmetric) under a subgroup $G\subset$ SO(3) if there is a set of Möbius transformation pairs $\{g,D_g\}$ with $g\in G$ acting on the domain S^2 and D_g acting on the target S^2, such that

$$R(g(z))=D_gR(z). \tag{I.4.15}$$

The transformations D_g should represent G in the sense that $D_{g_1}D_{g_2}=D_{g_1g_2}$. Both g and D_g will in practice be SU(2) matrices. For example, $g(z)$ can be expressed as $g(z)=(\alpha z+\beta)/(-\bar{\beta}z+\bar{\alpha})$ with $|\alpha|^2+|\beta|^2=1$. Replacing (α,β) by $(-\alpha,-\beta)$ has no effect, so g is effectively in SO(3). The same is true for D_g.

Some rational maps possess an additional symmetry of reflection or inversion. The transformation $z\mapsto\bar{z}$ is a reflection, whereas $z\mapsto-1/\bar{z}$ is the antipodal map on S^2, or inversion.

For $N=1$ the hedgehog map is $R(z)=z$. It is fully O(3) invariant, since $R(g(z))=g(z)$ for any $g\in$ SU(2) and $R(-1/\bar{z})=-1/\bar{R}(z)$. This map gives the standard exact hedgehog skyrmion solution (I.3.34) with the usual profile function $F(r)$.

A general map of degree two is of the form

$$R(z)=\frac{\alpha z^2+\beta z+\gamma}{\lambda z^2+\mu z+\nu}\,. \tag{I.4.16}$$

Lets impose the two $\mathbb{Z}_2$ symmetries $z\mapsto-z$ and $z\mapsto1/z$ which generate the viergruppe of 180° rotations about all three Cartesian axes. The conditions $R(-z)=R(z)$ and $R(1/z)=1/R(z)$ restrict R to the form

$$R(z)=\frac{z^2-a}{-az^2+1}\,. \tag{I.4.17}$$

By a target space Möbius transformation, we can bring a to lie in the interval $-1\le a\le1$, with the map degenerating at the endpoints. Further, a 90° rotation, $z\mapsto iz$, reverses the sign of a. The maps (I.4.17) have three reflection symmetries in the Cartesian axes, which are manifest when a is real. For example, $R(\bar{z})=\bar{R}(z)$ when a is real. A baryon density plot for this configuration is shown in Fig. 3a.

A subset of the degree three rational maps $N = 3$ has symmetry $\mathbb{Z}_2 \times \mathbb{Z}_2$, realized by the requirements $R(-z) = -R(z)$ and $R(1/z) = 1/R(z)$. The first condition implies that the numerator of R is even in z and the denominator is odd, or vice versa. Imposing the second condition as well gives maps of the form

$$R(z) = \frac{\sqrt{3}az^2 - 1}{z(z^2 - \sqrt{3}a)} \tag{I.4.18}$$

with a complex. The inclusion of the factor $\sqrt{3}$ is for a convenience. The parameter space of these maps should be thought of as a Riemann sphere with a complex coordinate a. The rational map degenerates for three values of a, namely $a = \infty$ and $a = \pm 1/\sqrt{3}$. A slightly subtler symmetry occurs if a is imaginary.

The Wronskian of maps having form (I.4.18) is

$$W(z) = -\sqrt{3}a(z^4 + \sqrt{3}(a - a^{-1})z^2 + 1). \tag{I.4.19}$$

Note that for $a = \pm\mathrm{i}$, W is proportional to a tetrahedral Klein polynomial [67]. If $a = 1$, W has a square symmetry, but the rational map does not have as much symmetry as this (see Fig. 3b).

The minimal energy skyrmion with $B = 4$ has octahedral symmetry, and there is a unique octahedrally symmetric $N = 4$ monopole. The octahedrally symmetric rational map of degree four can be embedded in a one parameter family of tetrahedrally symmetric maps

$$R(z) = c\frac{z^4 + 2\sqrt{3}\mathrm{i}z^2 + 1}{z^4 - 2\sqrt{3}\mathrm{i}z^2 + 1}, \tag{I.4.20}$$

where c is real. The numerator and the denominator are tetrahedrally symmetric Klein polynomials, so R is invariant up to a constant factor under any transformation in the tetrahedral group.

The Wronskian of the map (I.4.20) is proportional to $z(z^4 - 1)$ for all values of c. This is the face polynomial of a cube, with faces in the directions $0, 1, \mathrm{i}, -1, -\mathrm{i}, \infty$ (i.e. the directions of the Cartesian axes). It allows to understand why the baryon density vanishes in these directions, and hence why the skyrmion has a cubic shape, with its energy concentrated on the vertices and the edges of a cube (see Fig. 3c).

For $N = 5$ there is a family of rational maps with two real parameters, with the generic map having the D_{2d} symmetry, but having higher symmetry at special parameter values [68].

The family of these maps is

$$R(z) = \frac{z(z^4 + bz^2 + a)}{az^4 - bz^2 + 1} \tag{I.4.21}$$

with a and b real. The two generators of the D_{2d} symmetry are realized as $R(\mathrm{i}/\bar{z}) = \mathrm{i}/\bar{R}(z)$ and $R(-z) = -R(z)$.

The map $R(z) = z(z^4 - 5)/(-5z^4 + 1)$ has the Wronskian

$$W(z) = -5(z^8 + 14z^4 + 1), \tag{I.4.22}$$

which is proportional to the face polynomial of an octahedron. $N = 5$ baryon density plot is shown in Fig. 3d.

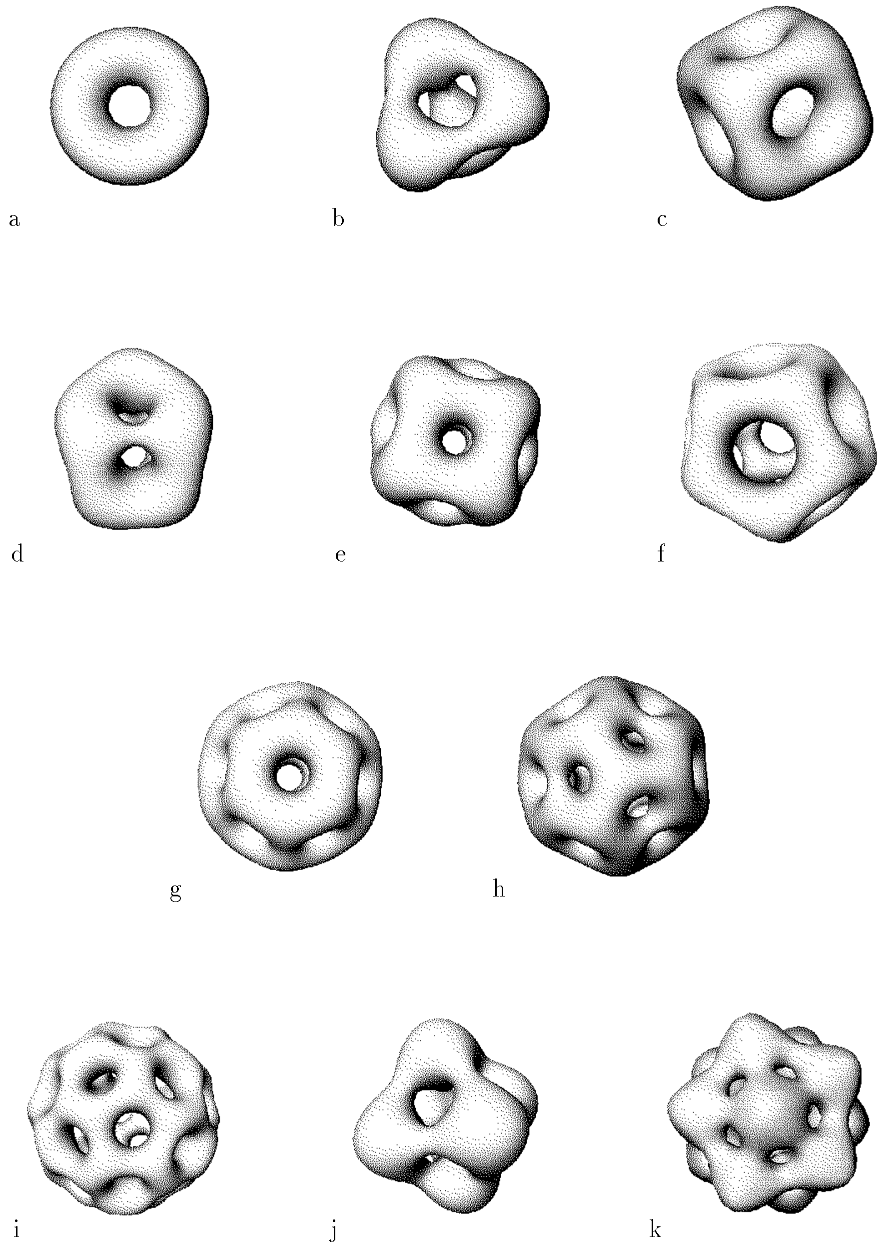

Figure 3: Surfaces of constant baryon density for the following Skyrme fields [49]: a) $B = 2$ torus, b) $B = 3$ tetrahedron, c) $B = 4$ cube, d) $B = 5$ with D_{2d} symmetry, e) $B = 6$ with D_{4d} symmetry, f) $B = 7$ dodecahedron, g) $B = 8$ with D_{6d} symmetry, h) $B = 9$ with tetrahedral symmetry, i) $B = 17$ buckyball, j) $B = 5$ octahedron, k) $B = 11$ icosahedron.

The skyrmions with $B = 6$ and $B = 8$ both have extended cyclic symmetry. For $B = 6$, the desired symmetry is D_{4d}. D_4 is generated by $z \mapsto \mathrm{i}z$ and $z \mapsto 1/z$. The rational maps

$$R(z) = \frac{z^4 + \mathrm{i}a}{z^2(\mathrm{i}az^4 + 1)} \tag{I.4.23}$$

have this symmetry, since $R(\mathrm{i}z) = -R(z)$ and $R(1/z) = 1/R(z)$. If a is real $R(\mathrm{e}^{\mathrm{i}\pi/4}\bar{z}) = \mathrm{i}\bar{R}(z)$ and the rational maps have D_{4d} symmetry. The Skyrme field has a polyhedral shape consisting of a ring of eight pentagons capped by squares above and below (see Fig. 3e).

For $B = 8$, the symmetry is D_{6d}. D_6 is generated by $z \mapsto \mathrm{e}^{\mathrm{i}\pi/3}z$ and $z \mapsto \mathrm{i}/z$. The rational maps

$$R(z) = \frac{z^6 - a}{z^2(az^6 + 1)} \tag{I.4.24}$$

have this symmetry. If a is real they have D_{6d} symmetry. The polyhedral shape is now a ring of twelve pentagons capped by hexagons above and below (see Fig. 3g).

The $N = 7$ case is similar to the cases $N = 6$ and $N = 8$, but the skyrmion has a dodecahedral shape (see Fig. 3f). A dodecahedron is a ring of ten pentagons capped by pentagons above and below.

Imposing the tetrahedral symmetry on degree nine maps they obtains the one real parameter family. The Skyrme field has a polyhedral shape consisting of four hexagons centered on the vertices of a tetrahedron, linked by four triples of pentagons (see Fig. 3h).

The skyrmion with $B = 11$ have the icosahedral symmetry. This icosahedral configuration is shown in Fig. 3k.

A highly symmetric configuration case is at $B = 17$, where it has been conjectured that the skyrmion has the icosahedrally symmetric, buckyball structure of carbon 60. The polyhedron does indeed have the buckyball form (see Fig. 3i), consisting of twelve pentagons, each surrounded by five hexagons, making a total of 32 polygons.

5. Generalization of the Skyrme model

By looking at the Skyrme model as a low energy effective theory from QCD in the limit in which the number of colours N_c is large, one finds that the Skyrme field takes values in SU(N_f), where N_f is the number of flavours of light quarks. In previous sections we have only considered the case of $N_\mathrm{f} = 2$, which is physically the most relevant since the up and down quarks are almost massless, and the SU(2) flavour symmetry between up and down quarks is only weakly broken in nature. A model with the SU(3) flavour symmetry, allowing for the strange quark, should have appropriate additional symmetry breaking terms to take into account the higher mass of the strange quark. This model is also a reasonable approximation. It allows to study the strange baryons and nuclei within the Skyrme model. The basic fields now describe pions, kaons, and the eta meson. There is still just one topological charge, identified as the baryon number. In the absence of any symmetry breaking mass terms, the three flavour Skyrme Lagrangian is given by the usual expression (I.3.21), however $U \in$ SU(3). There is also a Wess-Zumino term, which will be discussed in next section. It is important only in the quantization of skyrmions.

Solutions of the SU(3) model can be obtained by embedding the SU(2) skyrmions, and current evidence suggests that these are the minimal energy solutions at each charge. However, there are also solutions which do not correspond to the SU(2) embeddings. Although they have energies slightly higher than the embedded skyrmions, they are still low energy configurations. This symmetries are very different from the SU(2) solutions.

An example of a non-embedded solution is the dibaryon of Balachandran *et al.* [4], which is a spherically symmetric solution with $B = 2$. Explicitly, the Skyrme field is given by

$$U(\mathbf{x}) = \exp\left\{\mathrm{i}F_1(r)\mathbf{\Lambda}\cdot\hat{\mathbf{x}} + \mathrm{i}F_2(r)\left((\mathbf{\Lambda}\cdot\hat{\mathbf{x}})^2 - \frac{2}{3}\cdot\mathbb{1}_3\right)\right\}, \tag{I.5.25}$$

where $\mathbf{\Lambda}$ is a triplet of SU(3) matrices generating SO(3) and $F_1(r)$, $F_2(r)$ are real profile functions satisfying the boundary conditions $F_1(0) = F_2(0) = \pi$ and $F_1(\infty) = F_2(\infty) = 0$. Substituting this ansatz into the Skyrme Lagrangian density (I.3.21) leads to two coupled ordinary differential equations for $F_1(r)$ and $F_2(r)$. Solving these numerically yields an energy little higher than the energy of the embedded SU(2) rational map ansatz (I.4.17) of charge 2.

A different generalization of model is the Skyrme model constructed on a 3-sphere, in which the domain $\mathbb{R}_3$ is replaced by S^3_L, the 3-sphere of radius L, but the Skyrme field is still a map to the target space SU(2). The baryon number is the degree of U. This generalization has been studied in Ref. [69], and in a more geometrical context in Ref. [70]. By taking the limit $L \to \infty$ the Euclidean model is recovered, but it is possible to gain some additional understanding of skyrmions by first considering finite values of L.

Let μ, z be coordinates on S^3_L, with μ being the polar angle (the co-latitude) and z denoting the Riemann sphere coordinate on the 2-sphere at polar angle μ. Take F, R to be similar coordinates on the unit 3-sphere S^3_1, which we identify with the target manifold SU(2).

In general, a static field is given by functions $F(\mu, z, \bar{z})$ and $R(\mu, z, \bar{z})$. To find the $B = 1$ skyrmion we consider an analogue of the hedgehog field, an SO(3)-symmetric map of the form

$$F = F(\mu), \qquad R = z, \tag{I.5.26}$$

whose energy is

$$E = \frac{1}{3\pi}\int_0^\pi\left(L\sin^2\mu\left(F'^2 + \frac{2\sin^2 F}{\sin^2\mu}\right) + \frac{\sin^2 F}{L}\left(\frac{\sin^2 F}{\sin^2\mu} + 2F'^2\right)\right)\mathrm{d}\mu. \tag{I.5.27}$$

Among these maps there is the 1-parameter family of degree 1 conformal maps defined by

$$\tan\frac{F}{2} = \mathrm{e}^a\tan\frac{\mu}{2}, \tag{I.5.28}$$

where a is a real constant. These may be pictured as a stereographic projection from S^3_L to $\mathbb{R}^3$, followed by a rescaling by e^a, and then by an inverse stereographic projection from $\mathbb{R}^3$ to S^3_1. Substituting the expression (I.5.28) into the energy (I.5.27), and performing the integral gives

$$E = \frac{L}{1+\cosh a} + \frac{\cosh a}{2L}. \tag{I.5.29}$$

If $a = 0$ then (I.5.28) is the identity map with energy

$$E = \frac{1}{2}\left(L + \frac{1}{L}\right). \tag{I.5.30}$$

Note that if $L=1$ then $E=1$, so the Bogomolny bound is attained.

In the Euclidean limit $L\to\infty$ the radial variable should be identified as the combination $\tilde{r}=L\mu$, in which case the expression for the energy (I.5.27) takes the form of the standard energy expression of the classical Skyrme model (I.3.39). On a small 3-sphere the energy density of a $B=1$ skyrmion is uniformly distributed over S^3_L and the unbroken symmetry group is SO(4). However as the radius of the 3-sphere is increased beyond the critical value of $L=\sqrt{2}$ there is a bifurcation to a skyrmion localized around a point and the chiral symmetry is broken. Thus a phase transition occurs, when one moves from high to low baryon density, with a corresponding breaking of the chiral symmetry. This may have relevance to the physical issue of whether the quark confinement occurs at the same time as the chiral symmetry breaks when very dense quark matter becomes less dense.

If the charge $B>1$ the rational map ansatz can be applied again to produce low energy Skyrme fields which approximate the minimal energy skyrmions on S^3_L [71, 72], by taking $R(z)$ to be a degree B rational map and $F(\mu)$ is the associated energy minimizing profile function. This produces fields which tend to those of the Euclidean model as $L\to\infty$ and for all cases, this ansatz produces the good energy configurations.

The substitution of the field expressed in terms of pion fields (I.2.4) into the Lagrangian (I.3.21) reveals that the pions are massless. They are Goldstone bosons of the spontaneously broken chiraly symmetry. An additional term

$$L_{\text{mass}}=m_\pi^2\int \mathrm{Tr}\Big(U-\mathbb{1}\Big)\mathrm{d}^3x, \tag{I.5.31}$$

can be included in the Lagrangian of the Skyrme model to make the pions have a mass m_π. After the inclusion of (I.5.31) the skyrmion becomes exponentially localized, in contrast to the algebraic asymptotic behaviour of the Skyrme field in the massless pion model. This is because the modified equation of the hedgehog $F(\tilde{r})$ function,

$$\begin{aligned}F''(\tilde{r})\left(1+\frac{2\sin^2F(\tilde{r})}{\tilde{r}^2}\right)+F'^2(\tilde{r})\frac{\sin 2F(\tilde{r})}{\tilde{r}^2}+\frac{2}{\tilde{r}}F'(\tilde{r})\\ -\frac{\sin 2F(\tilde{r})}{\tilde{r}^2}-\frac{\sin 2F(\tilde{r})\sin^2F(\tilde{r})}{\tilde{r}^4}-m_\pi^2\tilde{r}\sin F=0.\end{aligned} \tag{I.5.32}$$

has the asymptotic Yukawa-type solution

$$F(\tilde{r})\sim\frac{A}{\tilde{r}}\mathrm{e}^{-m_\pi\tilde{r}}. \tag{I.5.33}$$

Clearly the energy of a single skyrmion with $m_\pi>0$ will be slightly higher than that with $m_\pi=0$, because the pion mass term is positive for all fields.

For higher charge skyrmions, the rational map approach works as before, but the profile function will again be slightly modified, leading to slightly higher energies.

6. The Wess-Zumino term

The three flavor QCD Lagrangian in the chiral limit ($m_u=m_d=m_s=0$) is globally invariant under $\mathrm{U(3)_L}\times\mathrm{U(3)_R}$. By Noether's theorem there are nine conserved vector and axial vector currents at the classical level [54]. Because of the Adler-Bell-Jackiw anomaly [73,74] the U(l)$_A$ symmetry is explicitly broken at the quantum level. So aside from the anomaly it is believed that the chiral symmetry is spontaneously broken through

$$\mathrm{U(3)_L} \otimes \mathrm{U(3)_R}/\mathrm{U(1)_A} \equiv \mathrm{SU(3)_L} \otimes \mathrm{SU(3)_R} \otimes \mathrm{U(1)_V} \rightarrow \mathrm{SU(3)_V} \otimes \mathrm{U(1)_V}, \tag{I.6.1}$$

with the appearance of eight massless Goldstone bosons, that corresponds to the pseudoscalar octet mesons: $\pi^0, \pi^\pm$, $\eta, K^0, \bar{K}^0, K^\pm$. In the spirit of the large N_c limit, the dynamics of the massless pseudoscalar mesons is dictated by a nonlinear σ-model Lagrangian such as (I.3.18) to leading order where $U(\mathbf{x})$ is an SU(3)-valued field of the form

$$U(\mathbf{x},t) = \exp \mathrm{i} \left(\lambda^a \frac{\pi^a(\mathbf{x},t)}{f_\pi} \right). \tag{I.6.2}$$

There the λ's are the ordinary Gell-Mann matrices with the normalization condition $\mathrm{Tr}\left(\lambda^a \lambda^b\right) = 2\delta_{a,b}$. Under $\mathrm{U(3)_L} \times \mathrm{U(3)_R}$, $U(\mathbf{x},t)$ transforms as follows:

$$\exp(\mathrm{i}Q_\mathrm{L}) U(\mathbf{x},t) \exp(-\mathrm{i}Q_\mathrm{R}), \tag{I.6.3}$$

where $Q_\mathrm{L,R}$ are the fundamental generators of U(3). Under parity, $U(\mathbf{x},t)$ transforms according to (I.3.38), respectively the pseudoscalar character of the octet mesons transforms:

$$\hat{\boldsymbol{\pi}}_\mathrm{op} \pi^a(\mathbf{x},t) \hat{\boldsymbol{\pi}}_\mathrm{op}^{-1} = \pi^a(-\mathbf{x},t). \tag{I.6.4}$$

Witten observed [75] that while the nonlinear σ-model (I.3.18) is invariant under global U(3)$_L \times$U(3)$_R$ and even under the QCD parity (I.3.38), it exhibits two discrete symmetries which are redundant with QCD namely

$$U(\mathbf{x},t) \rightarrow U(-\mathbf{x},t), \tag{I.6.5a}$$
$$U(\mathbf{x},t) \rightarrow U^\dagger(\mathbf{x},t). \tag{I.6.5b}$$

The latters forbid anomalous processes in which an even number of pseudoscalar mesons decay into an odd number and vice versa. As a remedy Witten proposed to modify the classical equations of motion in the nonlinear σ-model by adding explicitly a $\mathrm{U(3)_L} \otimes \mathrm{U(3)_R}$ invariant term that breaks the redundant symmetries (I.6.5a) and (I.6.5b) separately while preserving their combination i.e. the QCD parity operation (I.3.38). To break explicitly (I.6.5a) while maintaining Lorentz invariance requires the totally anti-symmetric Levi-Cevita tensor

$$\frac{1}{2} f_\pi^2 \partial^\mu R_\mu + \lambda \epsilon^{\mu\nu\alpha\beta} R_\mu R_\nu R_\alpha R_\beta = 0. \tag{I.6.6}$$

Under $x \rightarrow -x$, we have $\epsilon^{\mu\nu\alpha\beta} \rightarrow -\epsilon_{\mu\nu\alpha\beta}$, $\partial^\mu \rightarrow \partial_\mu$ and $R^\mu \rightarrow R_\mu$, hence

$$\frac{1}{2} f_\pi^2 \partial_\mu R^\mu - \lambda \epsilon_{\mu\nu\alpha\beta} R^\mu R^\nu R^\alpha R^\beta = 0. \tag{I.6.7}$$

Under $\pi^a \rightarrow -\pi^a$, we have $U(x) \rightarrow U^\dagger(x)$ and $R_\mu \rightarrow L_\mu = -U R_\mu U^\dagger$. Since $\partial^\mu L_\mu + U \partial^\mu R_\mu U^\dagger = 0$, we obtain

$$\frac{1}{2} f_\pi^2 \partial^\mu R_\mu + \lambda \epsilon^{\mu\nu\alpha\beta} R_\mu R_\nu R_\alpha R_\beta = 0. \tag{I.6.8}$$

In other words the redundant symmetries are lifted while their combination (QCD parity) is preserved.

To proceed to a quantum description starting from the classical field equations (I.6.6) we need the corresponding action functional. To do that is non trivial since the obvious candidate for the added term $\epsilon^{\mu\nu\alpha\beta}\,\mathrm{Tr}\,(R_\mu R_\nu R_\alpha R_\beta)$ vanishes identically in $(3+1)$ dimensions due to the cyclic property of the trace. In fact, the pertinent action functional involves the Wess-Zumino (WZ) term [44] of current algebra, and turns out to be non local in $(3+1)$ dimensions.

The solution of the problem raised by Witten [75] is suggested by the solution of a much simpler problem of an electron moving on the surface of a unit sphere surrounding a Dirac magnetic monopole [76] . The analogy of the preceding example with the $\mathrm{SU(3)_L} \otimes \mathrm{SU(3)_R}$ nonlinear σ-model is striking if we notice that for the vanishing magnetic field, the constrained equation on S^2 is invariant under $\mathbf{r} \to -\mathbf{r}$ and $t \to -t$ separately. The additional Lorentz force created by the magnetic monopole preserves only the combination $\mathbf{r} \to -\mathbf{r}$ and $t \to -t$. The Lorentz force is the analogue of the anomaly term in (I.6.6), while the geometrical analogue of the one-dimensional closed path S^1 on S^2 is a four-dimensional quasi-sphere $\mathrm{S}^3 \times \mathrm{S}^1$on $\mathrm{S}^3 \times \mathrm{S}^2$.

To elucidate these statements it is best to work in the Euclidean space with a compactified time direction, i.e. $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}^1 \to \mathbb{R}^3 \times \mathrm{S}^1$. Finite field configurations yield a compactification of $\mathbb{R}^3$ into S^3 and endow the space-time with the topology of a quasi sphere $\mathrm{S}^3 \times \mathrm{S}^1$. The latter can be thought of as the boundary of a five-dimensional manifold D_5

$$\mathrm{D}_5^+ = \mathrm{S}^3 \times \mathrm{S}^1 \times [0,1], \tag{I.6.9a}$$

$$\mathrm{D}_5^- = \mathrm{S}^3 \times \mathrm{S}^1 \times [-1,0], \tag{I.6.9b}$$

where we have used a decomposition of S^3. The SU(3) field $U(\mathbf{x})$ acts as a mapping from $\mathrm{S}^3 \times \mathrm{S}^1$ onto SU(3) whose group manifold is isomorphic to $\mathrm{S}^5 \times \mathrm{S}^3$ by Bott's theorem [77]. In analogy with the U(1) monopole where the action associated to the Lorentz force is a U(1) invariant on the boundaries $\mathrm{D}_2^\pm$, the action functional corresponding to the anomaly term in (I.6.6) should be sought as an $\mathrm{SU(3)_L} \times \mathrm{SU(3)_R}$ invariant on $\mathrm{D}_5^\pm$. To achieve this the SU(3) map $U(\mathbf{x})$ from $\mathrm{S}^3 \times \mathrm{S}^1$ onto SU(3) should be extended to a map $U(x)$ from $\mathrm{D}_5^\pm$ onto SU(3). Since

$$\left(\mathrm{S}^3 \times \mathrm{S}^2, \mathrm{S}^5 \times \mathrm{S}^3\right) \sim (\mathrm{S}^5, \mathrm{S}^5) \sim \pi_5(\mathrm{S}^5) \sim \mathbb{Z}, \tag{I.6.10}$$

by De-Rham's theorem [78] there must exist a topologically invariant and closed 5-form ω_5^0 on S^5, such that

$$\int_{\mathrm{S}^5} \omega_5^0 = \int_{\mathrm{S}^5} \mathrm{d}^5 x\, \mathcal{Q}_5^0 = 2\pi. \tag{I.6.11}$$

There $\mathcal{Q}_5^0$ is the Chern-Pontryagin density associated to $\pi_5(\mathrm{S}^5) \sim \mathbb{Z}$. To construct an explicit form of the pertinent isomorphism: $\pi^5(\mathrm{S}^5) \to \mathbb{Z}$, we can use a straightforward generalization of the proper construction that led to (I.3.21). In particular, we have

$$\mathcal{Q}_5^0 = \frac{\mathrm{i}}{240\pi^2}\epsilon^{0\mu\alpha\beta\gamma\delta}\,\mathrm{Tr}\Big(R_\mu R_\alpha R_\beta R_\gamma R_\delta\Big). \tag{I.6.12}$$

Its corresponding closed 5-form ω_5^0 is exact. The normalization in (I.6.12) is obtained by first using a polar parametrization of S^5 which yields $2\pi/5!A_5$, with $A_5 = \pi^3$ being the surface of

S^5 and then making the substitution $\phi^0 = 1$ and $\partial_\mu \phi^k \sim \mathrm{i} R^k_\mu$, $k = 1, 2, 3, 4, 5$ for any subset of SU(3). Indeed, if we define the 1-form $\alpha = R_\mu dx^\mu$, then

$$\omega^0_5 = \frac{\mathrm{i}}{240\pi^2} \mathrm{Tr}\Big(\alpha^5\Big), \tag{I.6.13}$$

where the wedge product has been omitted for convenience.

The Wess-Zumino Lagrangian associated to the anomaly term in (I.6.6) can be cast in the form

$$L_{\mathrm{WZ}} = +\lambda \int\limits_{\mathrm{D}_5^+} \omega^0_5 = -\lambda \int\limits_{\mathrm{D}_5^-} \omega^0_5, \qquad \lambda \in \mathbb{R}, \tag{I.6.14}$$

where $\mathrm{D}_5^\pm$ are the complementary discs defined in (I.6.9). To summarize:

- L_{WZ} is $\mathrm{SU(3)_L} \otimes \mathrm{SU(3)_R}$ invariant.
- L_{WZ} is topologically invariant, since ω^0_5 is closed.
- L_{WZ} depends only on the space-time boundary $\partial_\mu \mathrm{D}_5 = \mathrm{S}^3 \times \mathrm{S}^1$, since ω^0_5 is locally exact.

In terms of (I.6.14), the modified nonlinear σ-model action is

$$S_\pm = -\frac{f_\pi^2}{4} \mathrm{d}^4 x \, \mathrm{Tr}\Big(R^\mu R_\mu\Big) \pm \frac{\mathrm{i}\lambda}{240\pi^2} \int\limits_{\mathrm{D}_5^\pm} \mathrm{d}^5 x \epsilon^{\mu\alpha\beta\gamma\delta} \, \mathrm{Tr}\Big(R_\mu R_\alpha R_\beta R_\gamma R_\delta\Big). \tag{I.6.15}$$

The contour ambiguity in (I.6.15) can be resolved if one requires the generating functional (and hence the exponential factors $\exp(\mathrm{i}S_\pm)$) to be contour independent. This is fulfilled if and only if $\exp(\mathrm{i}S_+) = \exp(\mathrm{i}S_-)$, i.e.

$$\lambda\Bigg(\int\limits_{\mathrm{D}_5^+} \omega^0_5 + \int\limits_{\mathrm{D}_5^-} \omega^0_5\Bigg) = \lambda \int\limits_{\mathrm{D}_5^+ \cup \mathrm{D}_5^-} \omega^0_5 = \lambda \int\limits_{\mathrm{S}^3 \times \mathrm{S}^2} \omega^0_5 \equiv 2\pi\lambda \equiv 2\pi n, \quad \lambda = n = \text{integer}. \tag{I.6.16}$$

This shows that λ is topologically quantized. When analyzed in the context of QCD, this quantization is of fundamental relevance to the skyrmion.

The action (I.6.15) can be expresed in a non local form in the ordinary space-time. Indeed, to the leading order in the pseudoscalar fields $\phi(x) = \lambda^a \pi^a / f_\pi$, the Wess-Zumino term in (I.6.15) becomes

$$\begin{aligned} L_{\mathrm{WZ}} &= \pm \frac{n}{240\pi^2} \int\limits_{\mathrm{D}_5^\pm} \mathrm{d}^5 x \epsilon^{\mu\nu\alpha\beta\gamma} \, \mathrm{Tr}\Big(\partial_\mu\phi\partial_\nu\phi\partial_\alpha\phi\partial_\beta\phi\partial_\gamma\phi\Big) + ... \\ &= \frac{n}{240\pi^2} \int\limits_{\partial \mathrm{D}_5 = \mathrm{S}^3 \times \mathrm{S}^1} \mathrm{d}\Sigma_\mu \epsilon^{\mu\nu\alpha\beta\gamma} \, \mathrm{Tr}\Big(\partial_\mu\phi\partial_\nu\phi\partial_\alpha\phi\partial_\beta\phi\partial_\gamma\phi\Big) + ..., \end{aligned} \tag{I.6.17}$$

where Stoke's theorem was used. The expression (I.6.17) was originally investigated by Wess and Zumino [44] in the context of the effective Lagrangians.

The phenomenological implications of the topological quantization of the WZ term L_{WZ} are relevant to low energy observables. At low energy, it provides the canonical link between QCD and chiral effective descriptions based on the nonlinear σ-model (see I.2. section). In the context of the QCD, the number n is identified with the number of colors, i.e. $n \equiv N_\mathrm{c}$. Finally, the Wess-Zumino action can be written as

$$S_\mathrm{WZ}(U(\mathbf{x})) = -\frac{N_\mathrm{c}}{240\pi^2}\int\limits_{\mathrm{D}_5} \mathrm{d}^5x\epsilon^{\mu\nu\alpha\beta\gamma}\,\mathrm{Tr}\Big(R_\mu R_\nu R_\alpha R_\beta R_\gamma\Big). \tag{I.6.18}$$

In this equation it is understood that the pion field $U(\mathbf{x})$ is continued without singularities from the physical 4-dim space-time to a fifth dimension, so that the physical space-time serves as border of the 5-dim disk. Actually, it can be shown that the integrand in (I.6.18) is a full derivative, consequently the Wess-Zumino action (I.6.18) does not depend on a concrete way one continues the pion field inside the disk [79].

The Wess-Zumino term does not contribute to the classical energy, but it plays an important role in the quantum theory. Its introduction breaks the time reversal and parity symmetries of the model down to the combined symmetry operation: $t \to -t,\ x \to -x,\ U \to U^\dagger$, which appears to be realized in nature, unlike these individual symmetry operations.

7. Quantization of model

Quantization is a vital issue for skyrmions. It is more important than for the other solitons, because skyrmions are supposed to model physical baryons and nuclei, and a single baryon is a spin half fermion. Quantization of skyrmions may raise some questions, because the Skyrme model it is not a fundamental field theory, but rather a classical model that results from taking the limit of such a theory, including only some degrees of freedom of the original theory. Nevertheless, there is a rich experience from the nonrelativistic many-body problems, for example, from nuclear physics [80], suggesting the validity of such an approach for the study of collective properties at low energies.

To determine whether a skyrmion should be quantized as a fermion we can compare the amplitudes for the processes in which a skyrmion remains at rest for some long time t, and in which the skyrmion is slowly rotated through an angle 2π during this time. The sigma model and the Skyrme terms in the action do not distinguish between these two processes, since they involve two or more time derivatives. Since the Wess-Zumino term is only linear in time derivatives, it can distinguish them. In fact it results in the amplitudes for these two processes differing by a factor $(-1)^{N_\mathrm{c}}$, which shows that the skyrmion should be quantized as a fermion when N_c is odd, and in particular, in the physical case $N_\mathrm{c} = 3$ [3, 75].

For $N_\mathrm{c} = 2$ the above analysis does not apply, since the Wess-Zumino term vanishes for an SU(2)-valued field. To determine the appropriate quantization of an SU(2) skyrmion one may follow the approach of Finkelstein and Rubinstein [81], who showed that it is possible to quantize a soliton as a fermion by lifting the classical configuration space to its simply connected covering space. In the SU(2) Skyrme model, this is a double cover for any value of B.

A practical, approximate quantum theory of skyrmions is achieved by a rigid body quantization of the spin and isospin rotations. Vibrational modes whose excited states usually have considerably higher energy are ignored. For the $B = 1$ skyrmion, this quantization was carried out by Adkins, Nappi and Witten [5], who showed that the lowest energy states (compatible with the Finkelstein-Rubinstein constraints) have spin half and isospin half, and may be identified with the states of a proton or neutron.

The quantization of the $B = 2$ skyrmion was first discussed by Braaten and Carson [82], using a rigid body quantization. Their analysis was extended by Leese, Manton and Schroers [83], who also allowed the toroidal skyrmion to break up in the direction of the lowest vibrational mode, which corresponds to the attractive channel. Both calculations found that the lowest energy quantum state has isospin zero and spin 1, as expected for the deuteron. The second calculation gets closer to the usual physical picture of the deuteron as a rather loose proton-neutron bound state.

The quantization for higher charge skyrmions has been performed [82, 84–88] for charges $B \leq 8$, and gives the correct quantum numbers (spin, isospin and parity) for the experimentally observed ground states of nuclei in all cases except $B = 5$ and $B = 7$. A further study, making use of the topological properties of the space of rational maps, has allowed an extension of this analysis up to $B = 22$ [89]. The fact that some results do not agree with the experimental data is probably due to the restricted zero mode quantization, which does not allow any vibrational or deformation modes.

In our work we use the quantization in the "zero modes" or "collective coordinate" approach [5, 9]. As general, the construction of quantum theory passes three steps: the classical Lagrangian leads to the classical Hamiltonian which is modified to the quantum Hamiltonian. The quantization in the collective coordinate approach slightly modifies this sequence. It starts from the quantum Lagrangian from the outset. By the quantum Lagrangian we mean that the dynamical coordinates q^i and their time derivatives (velocities) $\dot{q}^i$ do not commute. The explicit commutation relations are not defined in the begining. These relations are extracted from the standard commutation relations $[q^i, p^j]$ after we pass to the quantum Hamiltonian (and define the canonical momenta p^j). It can be shown that this modified formalism [90–92] leads to a consistent quantum description.

The results of the modified and usual quantization sequences are usually different. Non-commuting quantum variables will generate additional terms while passing from the quantum Lagrangian to the quantum Hamiltonian. These terms are lost when we impose the canonical commutation relations after the Hamiltonian is obtained. This problem is similar to the operator ordering problem.

Following [93] we utilize the following detailed quantization sequence:

1. Introduce the quantum collective coordinates matrix $A\big(q(t)\big)$ (I.7.1). They are quantum in the sense that the dynamical variables $\mathbf{q}(t)$ and the time derivatives $\dot{\mathbf{q}}(t)$ do not commute.
2. Make the Lagrangian quantum.
3. Following the method described in [90–96] pass to the quantum Hamiltonian.
4. Solve the integro-differential equation for the quantum profile function $F(r)$.

Another important point is symmetry properties of the classical Lagrangian and the quantum Hamiltonian derived from it. There are quantization methods (for example, the general covariant Hamiltonian method [97]) preserving original classical Lagrangian symmetries. The symmetric Weyl ordering of the operators $\mathbf{q}$ and $\mathbf{p}$ (used in the work), however, cannot avoid a risk that the quantum Hamiltonian has a chiral symmetry breaking term [98] .

According to G.S. Adkins *et al.* [5] we employ the collective rotational coordinates to separate the variables which depend on the time and spatial coordinates

$$U\big(\mathbf{x}, q(t)\big) = A\big(q(t)\big) U_{\text{cl}}(\mathbf{x}) A^{\dagger}\big(q(t)\big), \qquad A\big(q(t)\big) \in \mathrm{SU}(N_c). \tag{I.7.1}$$

The real independent parameters $q(t) = \big(q^1(t), q^2(t), ..., q^k(t)\big)$, where $k = \dim(\mathrm{SU}(N_c))$, are the dynamical quantum variables – the skyrmion rotation (Euler) angles in the internal space

$\mathrm{SU}(N_\mathrm{c}) = \mathrm{diag}\big(\mathrm{SU}(N_\mathrm{c})_\mathrm{L} \otimes \mathrm{SU}(N_\mathrm{c})_\mathrm{R}\big)$ but not in the geometric space. The skyrmion remains static in the geometric space.

Quantum fluctuations near the classical solution can be put into two different classes. Namely, the fluctuation modes which are generated by the action or the Hamiltonian symmetries and the modes orthogonal to the symmetric one. Symmetric fluctuation modes are of primary importance in the quantum description because the infinitely small energy perturbation can lead to reasonable deviations from the classical solution. As a consequence, the collective rotation matrices $A(t)$ and $A^\dagger(t)$ in (I.7.1) are not required to be small (i.e. close to the identity matrix) [62].

We shall consider the Skyrme Lagrangian quantum mechanically *ab initio* [9,93]. The generalized coordinates $q(t)$ and velocities $\dot{q}(t)$ then satisfy the commutation relations

$$\big[\dot{q}^k,\, q^l\big] = -\mathrm{i}g^{kl}(q). \tag{I.7.2}$$

Here the tensor $g^{kl}(q)$ is a function of the generalized coordinates q only. Its explicit form is determined after the quantization condition has been imposed. The tensor g^{kl} is symmetric with respect to an interchange of the indices k and l as a consequence of the commutation relation $[q^k,\, q^l] = 0$. Indeed, differentiation of the relation gives $[\dot{q}^k,\, q^l] = [\dot{q}^l,\, q^k]$, from what it follows that g^{kl} is symmetric. The commutation relation between a generalized velocity component $\dot{q}^k$ and an arbitrary function $G(q)$ is given by

$$\big[\dot{q}^k,\, G(q)\big] = -\mathrm{i}\sum_r g^{kl}(q)\frac{\partial}{\partial q^l}G(q). \tag{I.7.3}$$

We shall employ the Weyl ordering for the noncommuting operators $\dot{q}$ and $G(q)$ through

$$(\partial_t G(q))_\mathrm{W} = \frac{1}{2}\Big\{\dot{q}^l,\, \frac{\partial G(q)}{\partial q^l}\Big\}, \tag{I.7.4}$$

which is a consequence of application of the Newton-Leibnitz rule to the Taylor series expansion of the arbitrary function $G(q)$:

$$G(q) = G(q_0) + G'(q)\Big|_{q=q_0} q + \frac{1}{2}G''(q)\Big|_{q=q_0} q^2 + \cdots . \tag{I.7.5}$$

Indeed,

$$(\partial_t q^2)_\mathrm{W} = \partial_t(q\,q) = \dot{q}q + q\dot{q} = \frac{1}{2}\Big\{\dot{q},\, \frac{\mathrm{d}(q^2)}{\mathrm{d}q}\Big\}, \tag{I.7.6a}$$

$$(\partial_t q^3)_\mathrm{W} = \partial_t(q\,q\,q) = \dot{q}q^2 + q\dot{q}q + q^2\dot{q} = \frac{3}{2}(\dot{q}q^2 + q^2\dot{q}) = \frac{1}{2}\Big\{\dot{q},\, \frac{\mathrm{d}(q^3)}{\mathrm{d}q}\Big\}, \tag{I.7.6b}$$

....

$$(\partial_t q^n)_\mathrm{W} = \partial_t\underbrace{(q\ldots q)}_{n} = \dot{q}q^{n-1} + \cdots + q^{n-1}\dot{q} = \frac{1}{2}\Big\{\dot{q},\, \frac{\mathrm{d}q^n}{\mathrm{d}q}\Big\}. \tag{I.7.6c}$$

The operator ordering is fixed by the form of the classical Lagrangian and no further ambiguity associated with it appears at the level of the Hamiltonian. In order to find the explicit form of $g^{kl}(q)$ one can substitute (I.7.1) into the expression of the Lagrangian density (I.3.21) and keep only the terms quadratic in velocities

$$\begin{aligned}\hat{L}(\dot{q},q,F) &= \frac{1}{N}\int \hat{\mathcal{L}}\big(x,\dot{q}(t),q(t),F(r)\big)r^2\mathrm{d}^3r \\ &= \frac{1}{2}\dot{q}^k g_{kl}(q)\dot{q}^l + \big[(\dot{q})^0\text{– order term}\big]\,, \end{aligned} \tag{I.7.7}$$

where a normalization factor N is introduced in the Lagrangian in order to ensure the baryon number 1 for the spherically symmetric skyrmion case. After the determination of the metric tensor $g_{kl}(q)$, we find the soliton moments of inertia and the explicit expression of the collective coordinates $A^\dagger\dot{A}$. By using these expressions and a set of auxiliary expressions which shown in appendices A, B and C, the complete expression of the Skyrme model Lagrangian density is obtained. Later special techniques are applied to construct the effective Hamiltonian.

II. Canonical quantization of the SU(3) Skyrme model in a general representation

Both the SU(2) and SU(3) Skyrme versions of the model have been quantized canonically in Refs. [9, 93, 99] in the collective coordinate formalism. The canonical quantization procedure leads to quantum corrections to the skyrmion mass, which restore the stability of the soliton solutions that is lost in the semiclassical quantization. This method has subsequently been generalized to the unitary fields $U(\mathbf{x},t)$ that belong to the general representations of the SU(2) [11–13], along with a demonstration that the quantum corrections, which stabilize the soliton solutions, are representation dependent.

The aim of this chapter is to extend the canonically quantized Skyrme model to the general irreducible representations (irrep) of SU(3). The motivation is the absence of any *a priori* reason to restrict the collective chiral models to the fundamental representation of the group. The focus here is on the mathematical aspects of the model, and on the derivation of both the Hamiltonian density and the Hamiltonian, in order to elucidate their representation dependence. The possible phenomenological applications both in hyperon and hypernuclear phenomenology as well as in the Skyrme model description of the quantum Hall effect [6] and Bose-Einstein condensates [7], are not elaborated.

Similarly to the SU(2) case, the solutions to the SU(3) Skyrme model depend in an essential way on the dimension. Remarkably the Wess-Zumino term vanishes in all self-adjoint irreps of SU(3), as it is proportional to the cubic Casimir operator $\hat{C}_3^{\mathrm{SU(3)}}$ in the collective coordinate method of separation of the dependence on space and time variables. In the self adjoint irreps the symmetry breaking mass term in the model reduces to the SU(2) form.

1. Definitions for the unitary SU(3) soliton field

The unitary field $U(\mathbf{x},t)$ is defined for general irreps (λ,μ) of SU(3) in addition to the fundamental representation $(1,0)$. The related Young tableaux are denoted $[\lambda_1,\lambda_2,\lambda_3]$, where $\lambda=\lambda_1-\lambda_2$, $\mu=\lambda_2-\lambda_3$. A group element is specified by eight real parameters $\alpha^i(\mathbf{x},t)$. The unitary field is expressed in the form of the Wigner D matrices for SU(3) in (λ,μ) irrep as

$$U(\mathbf{x},t)=D^{(\lambda,\mu)}(\alpha^i(\mathbf{x},t)). \tag{II.1.1}$$

The one-form of the unitary field belongs to the Lie algebra of SU(3). The one-forms may be determined by the functions $C_i^{(Z,I,M)}(\alpha)$ and $C_i'^{(Z,I,M)}(\alpha)$. Their explicit expressions depend on the specific group parametrization

$$\partial_i U U^\dagger = \left(\frac{\partial}{\partial\alpha^i}U\right)U^\dagger = C_i^{(Z,I,M)}(\alpha)\left\langle\ \middle|\ J^{(1,1)}_{(Z,I,M)}\middle|\ \right\rangle,$$
$$U^\dagger\partial_i U = U^\dagger\frac{\partial}{\partial\alpha^i}U = C_i'^{(Z,I,M)}(\alpha)\left\langle\ \middle|\ J^{(1,1)}_{(Z,I,M)}\middle|\ \right\rangle, \tag{II.1.2}$$

The basis states of irrep (1,1) are specified by the parameters isospin I, and its projections M and Z, which is related to the hypercharge as $Y = -2Z$.

The parametrization for the SU(3) model, and the expressions of the differential Casimir operator in terms of the Euler angles, have been proposed by Yabu and Ando [100]. In contrast to their approach, for simplicity, we do not determine the specific parametrization and use the general properties of the functions $C_i^{(Z,I,M)}(\alpha)$ (see B.1). The SU(3) generators are defined as components of the irreducible tensors $(1,1)$ and may be expanded in terms of the Gell-Man generators Λ_k:

$$\begin{aligned}
&\Lambda_1 = \sqrt{2}\big(J^{(1,1)}_{0,1,-1} - J^{(1,1)}_{0,1,1}\big); && J^{(1,1)}_{0,1,1} = -\frac{1}{2\sqrt{2}}(\Lambda_1 + \mathrm{i}\Lambda_2);\\
&\Lambda_2 = \mathrm{i}\sqrt{2}\big(J^{(1,1)}_{0,1,-1} + J^{(1,1)}_{0,1,1}\big); && J^{(1,1)}_{0,1,-1} = \frac{1}{2\sqrt{2}}(\Lambda_1 - \mathrm{i}\Lambda_2);\\
&\Lambda_3 = 2J^{(1,1)}_{0,1,0}; && J^{(1,1)}_{0,1,0} = \frac{1}{2}\Lambda_3;\\
&\Lambda_4 = \sqrt{2}\big(J^{(1,1)}_{-\frac12,\frac12,\frac12} + J^{(1,1)}_{\frac12,\frac12,-\frac12}\big); && J^{(1,1)}_{-\frac12,\frac12,\frac12} = \frac{1}{2\sqrt{2}}(\Lambda_4 + \mathrm{i}\Lambda_5);\\
&\Lambda_5 = -\mathrm{i}\sqrt{2}\big(J^{(1,1)}_{-\frac12,\frac12,\frac12} - J^{(1,1)}_{\frac12,\frac12,-\frac12}\big); && J^{(1,1)}_{\frac12,\frac12,-\frac12} = \frac{1}{2\sqrt{2}}(\Lambda_4 - \mathrm{i}\Lambda_5);\\
&\Lambda_6 = \sqrt{2}\big(J^{(1,1)}_{-\frac12,\frac12,-\frac12} - J^{(1,1)}_{\frac12,\frac12,\frac12}\big); && J^{(1,1)}_{-\frac12,\frac12,-\frac12} = \frac{1}{2\sqrt{2}}(\Lambda_6 + \mathrm{i}\Lambda_7);\\
&\Lambda_7 = -\mathrm{i}\sqrt{2}\big(J^{(1,1)}_{-\frac12,\frac12,-\frac12} + J^{(1,1)}_{\frac12,\frac12,\frac12}\big); && J^{(1,1)}_{\frac12,\frac12,\frac12} = -\frac{1}{2\sqrt{2}}(\Lambda_6 - \mathrm{i}\Lambda_7);\\
&\Lambda_8 = -2J^{(1,1)}_{0,0,0}; && J^{(1,1)}_{0,0,0} = -\frac{1}{2}\Lambda_8.
\end{aligned} \tag{II.1.3}$$

The Λ_i matrices are hermitian $\Lambda_i^\dagger = \Lambda_i$. Although the generators (II.1.3) are non-hermitian $\big(J^{(1,1)}_{(Z,I,M)}\big)^\dagger = (-1)^{Z+M}J^{(1,1)}_{(-Z,I,-M)}$, their commutation relations nevertheless have a simple form

$$\left[J^{(1,1)}_{(Z',I',M')}, J^{(1,1)}_{(Z'',I'',M'')}\right] = -\sum_{(Z,I,M)}\sqrt{3}\begin{bmatrix}(1,1) & (1,1) & (1,1)_a\\ (Z',I',M') & (Z'',I'',M'') & (Z,I,M)\end{bmatrix} J^{(1,1)}_{(Z,I,M)}\,. \tag{II.1.4}$$

For the specification of the basis states in a general irrep (λ,μ) the parameters (z,j,m) are employed. The hypercharge is $y = \frac{2}{3}(\mu-\lambda) - 2z$. The basis state parameters satisfy these inequalities:

$$\begin{aligned}
j - m &\geq 0\,, & j - z &\geq 0\,,\\
j + m &\geq 0\,, & j + z &\geq 0\,,\\
\lambda + z - j &\geq 0\,, & \mu - z - j &\geq 0\,,
\end{aligned} \tag{II.1.5}$$

where the values on the left hand sides are integers. The generators (II.1.3) act on the basis states as follows:

$$J^{(1,1)}_{(0,0,0)}\left|\begin{matrix}(\lambda,\mu)\\ z,j,m\end{matrix}\right\rangle = -\frac{\sqrt{3}}{2}y\left|\begin{matrix}(\lambda,\mu)\\ z,j,m\end{matrix}\right\rangle, \quad J^{(1,1)}_{(0,1,-1)}\left|\begin{matrix}(\lambda,\mu)\\ z,j,m\end{matrix}\right\rangle = \sqrt{\tfrac{1}{2}(j+m)(j-m+1)}\left|\begin{matrix}(\lambda,\mu)\\ z,j,m-1\end{matrix}\right\rangle,$$
$$J^{(1,1)}_{(0,1,0)}\left|\begin{matrix}(\lambda,\mu)\\ z,j,m\end{matrix}\right\rangle = m\left|\begin{matrix}(\lambda,\mu)\\ z,j,m\end{matrix}\right\rangle, \quad J^{(1,1)}_{(0,1,1)}\left|\begin{matrix}(\lambda,\mu)\\ z,j,m\end{matrix}\right\rangle = -\sqrt{\tfrac{1}{2}(j-m)(j+m+1)}\left|\begin{matrix}(\lambda,\mu)\\ z,j,m+1\end{matrix}\right\rangle,$$

$$\begin{aligned}
J^{(1,1)}_{(-\frac{1}{2},\frac{1}{2},\frac{1}{2})}\left|\begin{matrix}(\lambda,\mu)\\ z,j,m\end{matrix}\right\rangle &= \sqrt{\tfrac{(\lambda+z-j)(\mu-z+j+2)(j-z+1)(j+m+1)}{2(2j+1)(2j+2)}}\left|\begin{matrix}(\lambda,\mu)\\ z-\frac{1}{2},j+\frac{1}{2},m+\frac{1}{2}\end{matrix}\right\rangle \\
&\quad - \sqrt{\tfrac{(\lambda+z+j+1)(\mu-z-j+1)(j+z)(j-m)}{4j(2j+1)}}\left|\begin{matrix}(\lambda,\mu)\\ z-\frac{1}{2},j-\frac{1}{2},m+\frac{1}{2}\end{matrix}\right\rangle, \\
J^{(1,1)}_{(\frac{1}{2},\frac{1}{2},-\frac{1}{2})}\left|\begin{matrix}(\lambda,\mu)\\ z,j,m\end{matrix}\right\rangle &= -\sqrt{\tfrac{(\lambda+z+j+2)(\mu-z-j)(z+j+1)(j-m+1)}{2(2j+1)(2j+2)}}\left|\begin{matrix}(\lambda,\mu)\\ z+\frac{1}{2},j+\frac{1}{2},m-\frac{1}{2}\end{matrix}\right\rangle \\
&\quad + \sqrt{\tfrac{(\lambda+z-j+1)(\mu-z+j+1)(j-z)(j+m)}{4j(2j+1)}}\left|\begin{matrix}(\lambda,\mu)\\ z+\frac{1}{2},j-\frac{1}{2},m-\frac{1}{2}\end{matrix}\right\rangle, \\
J^{(1,1)}_{(-\frac{1}{2},\frac{1}{2},-\frac{1}{2})}\left|\begin{matrix}(\lambda,\mu)\\ z,j,m\end{matrix}\right\rangle &= \sqrt{\tfrac{(\lambda+z-j)(\mu-z+j+2)(j-z+1)(j-m+1)}{2(2j+1)(2j+2)}}\left|\begin{matrix}(\lambda,\mu)\\ z-\frac{1}{2},j+\frac{1}{2},m-\frac{1}{2}\end{matrix}\right\rangle \\
&\quad + \sqrt{\tfrac{(\lambda+z+j+1)(\mu-z-j+1)(j+z)(j+m)}{4j(2j+1)}}\left|\begin{matrix}(\lambda,\mu)\\ z-\frac{1}{2},j-\frac{1}{2},m-\frac{1}{2}\end{matrix}\right\rangle, \\
J^{(1,1)}_{(\frac{1}{2},\frac{1}{2},\frac{1}{2})}\left|\begin{matrix}(\lambda,\mu)\\ z,j,m\end{matrix}\right\rangle &= -\sqrt{\tfrac{(\lambda+z+j+2)(\mu-z-j)(j+z+1)(j+m+1)}{2(2j+1)(2j+2)}}\left|\begin{matrix}(\lambda,\mu)\\ z+\frac{1}{2},j+\frac{1}{2},m+\frac{1}{2}\end{matrix}\right\rangle \\
&\quad - \sqrt{\tfrac{(\lambda+z-j+1)(\mu-z+j+1)(j-z)(j-m)}{4j(2j+1)}}\left|\begin{matrix}(\lambda,\mu)\\ z+\frac{1}{2},j-\frac{1}{2},m+\frac{1}{2}\end{matrix}\right\rangle.
\end{aligned} \tag{II.1.6}$$

The basis states are chosen such that the generators $J^{(1,1)}_{(0,1,0)}$ and $J^{(1,1)}_{(0,0,0)}$, as well as the Casimir operator of the SU(2) subgroup $\hat{C}^{\mathrm{SU(2)}} = \sum(-1)^M J^{(1,1)}_{(0,1,M)} J^{(1,1)}_{(0,1,-M)}$, are diagonal and thus provide a labelling of the basis states

$$\hat{C}^{\mathrm{SU(2)}}\left|\begin{matrix}(\lambda,\mu)\\ z,j,m\end{matrix}\right\rangle = j(j+1)\left|\begin{matrix}(\lambda,\mu)\\ z,j,m\end{matrix}\right\rangle. \tag{II.1.7}$$

The calculation method of the Clebsch-Gordan (CG) coefficients of SU(3), which were used for derivation of the SU(3) Skyrme model in a general representation, was used from the work of J.G. Kuriyan *et al.* [101]. An algebraic tabulation of the CG coefficients which occur in the reduction of the direct product $(\lambda,\mu)\otimes(1,1)$ into irreducible representations is made. The phase convention is an explicitly stated generalization of the well-known Condon and Shortley phase convention for SU(3).

The CG coefficients factorize [102, 103] according to

$$\begin{bmatrix} (1,1) & (1,1) & (1,1)_\gamma \\ Z',I',M' & Z'',I'',M'' & Z,I,M \end{bmatrix} = \begin{bmatrix} (1,1) & (1,1) & (1,1)_\gamma \\ (Y')Z',I' & (Y'')Z'',I'' & (Y)Z,I \end{bmatrix}\begin{bmatrix} I' & I'' & I \\ M' & M'' & M \end{bmatrix}, \tag{II.1.8}$$

where the second factor on right hand side of the equation refers to the isospin SU(2) subgroup of SU(3), and the first factor (isoscalar factor) is independent of M', M'' and M. The label γ in the CG coefficient takes two values $\gamma = 1$ and 2. The first value denotes the antisymmetrical CG coefficient (sometimes it is labelled by index a) while the second value denotes the symmetrical CG coefficient (sometimes it is labelled by the index s). Hence, the CG coefficient of SU(2) being well-known, we need to tabulate only the isoscalar factors. Some isoscalar factors both for the symmetrical and antisymmetrical representation, and their properties are shown in Appendix B.

2. The Classical SU(3) Skyrme model

The action of the Skyrme model in SU(3) is taken to have the form

$$S = \int \mathrm{d}^4 x (\mathcal{L}_{\text{Sk}} + \mathcal{L}_{\text{SB}}) + S_{\text{WZ}}\,, \tag{II.2.1}$$

where $\mathcal{L}_{\text{Sk}}$ is the chirally symmetric Lagrangian density (I.3.21) in which the right chiral current is defined as

$$R_\mu = (\partial_\mu U)\, U^\dagger = \partial_\mu \alpha^i C_i^{(Z,I,M)}(\alpha) \left\langle \;\middle|\, J^{(1,1)}_{(Z,I,M)} \,\middle|\; \right\rangle. \tag{II.2.2}$$

The Greek characters indicate differentiation with respect to the spacetime variables $\partial_\mu \equiv \partial/\partial x^\mu$ in the metric $\operatorname{diag}(\eta_{\alpha\beta}) = (1,-1,-1,-1)$. The symmetry breaking term $\mathcal{L}_{\text{SB}}$ and the Wess-Zumino action S_{WZ} are specified below.

Upon substitution of (II.2.2) into the Skyrme Lagrangian (I.3.21) the classical Lagrangian density may be expressed in terms of the group parameters α^i as

$$\begin{aligned}
\mathcal{L}_{\text{Sk}} = \frac{3}{32N} \dim(\lambda,\mu) C_2^{\text{SU(3)}}(\lambda,\mu) \Bigg\{ & -f_\pi^2 (-1)^A \partial_\mu \alpha^i C_i^{(A)}(\alpha) \partial^\mu \alpha^{i'} C_{i'}^{(-A)}(\alpha) \\
& + \frac{3}{8e^2} (-1)^A \partial_\mu \alpha^i C_i^{(A^1)}(\alpha) \partial_\nu \alpha^{i'} C_{i'}^{(A^2)}(\alpha) \\
& \times \partial^\mu \alpha^{i''} C_{i''}^{(A^3)}(\alpha) \partial^\nu \alpha^{i'''} C_{i'''}^{(A^4)}(\alpha) \begin{bmatrix} (1,1) & (1,1) & (1,1)_a \\ (A^1) & (A^2) & (A) \end{bmatrix} \begin{bmatrix} (1,1) & (1,1) & (1,1)_a \\ (A^3) & (A^4) & (-A) \end{bmatrix} \Bigg\}.
\end{aligned} \tag{II.2.3}$$

At the end of equation (II.2.3), only the SU(3) Clebsch-Gordan coefficients with the antisymmetric irrep coupling are included and there is no summation over the irrep multiplicity. The capital Latin character indices (A) denote the state label (Z,I,M). $(-A)$ denotes $(-Z,I,-M)$ and $(-1)^A = (-1)^{Z+M}$. The dependence on the group irrep appears as an overall factor because of the calculation of trace (B.17) where the dimension of the irrep and the quadratic Casimir operator of SU(3) are included.

The time component of the conserved topological current in the Skyrme model represents the baryon number density which in terms of the variables $\alpha^i(x,t)$ takes the form

$$\begin{aligned}
\mathcal{B}^0(x) &= \frac{1}{24\pi^2 N} \epsilon^{0ikl} \operatorname{Tr}\Big((\partial_i U)\, U^\dagger\, (\partial_k U)\, U^\dagger\, (\partial_l U)\, U^\dagger \Big) \\
&= \frac{(-1)^A}{2^7 \sqrt{3} \pi^2 N} \dim(\lambda,\mu) C_2^{\text{SU(3)}}(\lambda,\mu) \epsilon^{abc} \partial_a \alpha^i C_i^{(A)}(\alpha) \\
&\quad \times \partial_b \alpha^{i'} C_{i'}^{(A')}(\alpha) \partial_c \alpha^{i''} C_{i''}^{(A'')}(\alpha) \begin{bmatrix} (1,1) & (1,1) & (1,1)_a \\ (A') & (A'') & (-A) \end{bmatrix}.
\end{aligned} \tag{II.2.4}$$

For the classical chiral symmetric Skyrme model the dependence on the irrep is contained in the overall factor N. The normalization factor

$$N = \frac{1}{4}\dim(\lambda,\mu) C_2^{\mathrm{SU(3)}}(\lambda,\mu)\,, \tag{II.2.5}$$

is chosen requiring that the smallest non trivial baryon number equals unity: $B = \int \mathrm{d}^3 x \mathcal{B}^0(x) = 1$. The dynamics of the system is independent of the overall factor in the Lagrangian. Therefore in the classical case the Skyrme model defined in an arbitrary irrep is equivalent to the Skyrme model in the fundamental representation $(1,0)$, for which $N = 1$.

The classical soliton solution of the hedgehog type for the (λ,μ) irrep of the SU(3) group may be expressed as a direct sum of the hedgehog ansatz in the SU(2) irreps [12]. The SU(2) representations embedded in the (λ,μ) irrep are defined by the canonical SU(3) $\supset$ SU(2) chain. The hedgehog generalization takes the form

$$\exp \mathrm{i}\big(\boldsymbol{\tau}\cdot\hat{x}\big)F(r) \to U_0\big(\hat{x}, F(r)\big) = \exp \mathrm{i}2\Big(J^{(1,1)}_{(0,1,\cdot)}\cdot\hat{x}\Big)F(r) = \sum_{z,j}^{(\lambda,\mu)} \oplus D^j(\alpha)\,, \tag{II.2.6}$$

where $\boldsymbol{\tau}$ are the Pauli matrices and $\hat{x}$ is the unit vector. The expressions of the Euler angles α of the SU(2) subgroup are shown in (A.12). The normalization factor (II.2.5) ensures that the baryon number density for the hedgehog skyrmion in a general irrep has the usual form

$$\begin{aligned}\mathcal{B}^0(x) &= \frac{1}{24\pi^2 N}\,\epsilon^{0ikl}\,\mathrm{Tr}\Big(\left(\partial_i U_0\right)U_0^\dagger\left(\partial_k U_0\right)U_0^\dagger\left(\partial_l U_0\right)U_0^\dagger\Big)\\ &= -\,\frac{1}{2\pi^2}\frac{\sin^2 F(r)}{r^2}F'(r)\,.\end{aligned} \tag{II.2.7}$$

With the hedgehog ansatz (II.2.6), and after the renormalization with the factor (II.2.5), the Lagrangian density (I.3.21) reduces to the following simple form

$$\begin{aligned}\mathcal{L}_{\mathrm{cl}}\big(F(r)\big) = -\,\mathcal{M}_{\mathrm{cl}}(F(r)) = -\bigg\{&\frac{f_\pi^2}{2}\Big(F'^2 + \frac{2}{r^2}\sin^2 F\Big)\\ &+ \frac{1}{2e^2}\frac{\sin^2 F}{r^2}\Big(2F'^2 + \frac{\sin^2 F}{r^2}\Big)\bigg\}\,.\end{aligned} \tag{II.2.8}$$

Variation of the classical hedgehog soliton mass leads to standard differential equation for the profile function $F(r)$.

The SU(3) chiral symmetry breaking term of the Lagrangian density is defined here as

$$\mathcal{L}_{\mathrm{SB}} = -\mathcal{M}_{\mathrm{SB}} = -\frac{1}{N}\frac{f_\pi^2}{4}\bigg(m_0^2\,\mathrm{Tr}\Big(U + U^\dagger - 2\cdot\mathbb{1}\Big) - 2m_8^2\,\mathrm{Tr}\Big(\big(U+U^\dagger\big)\,J^{(1,1)}_{(0,0,0)}\Big)\bigg). \tag{II.2.9}$$

This form is chosen so that it reduces to the mass term of the π, K, η mesons when the unitary field $U(\mathbf{x},t) = \exp\{\frac{\mathrm{i}}{f_\pi}\varphi_k\Lambda_k\}$ is expanded around the classical vacuum $U = \mathbb{1}$:

$$\mathcal{L}_{\mathrm{SB}} = -\frac{1}{2}m_\pi^2\big(\varphi_1^2+\varphi_2^2+\varphi_3^2\big) - \frac{1}{2}m_K^2\big(\varphi_4^2+\varphi_5^2+\varphi_6^2+\varphi_7^2\big) - \frac{1}{2}m_\eta^2\varphi_8^2 + \dots\,. \tag{II.2.10}$$

For arbitrary irrep the coefficients in the symmetry breaking term can be readily obtained as

$$m_0^2 = \frac{1}{3}\big(m_\pi^2 + 2m_K^2\big), \qquad m_8^2 = \frac{10}{3\sqrt{3}}\frac{C_2^{\mathrm{SU(3)}}(\lambda,\mu)}{C_3^{\mathrm{SU(3)}}(\lambda,\mu)}\big(m_\pi^2 - m_K^2\big), \tag{II.2.11}$$

where

$$C_3^{\mathrm{SU(3)}}(\lambda,\mu)=\frac{1}{9}(\lambda-\mu)(2\lambda+\mu+3)(2\mu+\lambda+3)\,, \tag{II.2.12}$$

is the eigenvalue of the cubic Casimir operator of SU(3).

For the self adjoint irreps $\lambda=\mu$ the symmetry breaking part of the Lagrangian (II.2.9) is proportional to $m_0^2=\frac{1}{4}m_\pi^2$ only. The Gell-Mann-Okubo mass formula

$$m_\pi^2+3m_\eta^2-4m_K^2=0\,, \tag{II.2.13}$$

is satisfied in all but the self adjoint irreps.

3. Quantization of the skyrmion

The direct quantization of the Skyrme model leads to rather complicated equations [11] even in the case of SU(2). Here the collective coordinates for the unitary field U in the (λ,μ) irrep are employed for the separation of the variables, which depend on the temporal and spatial coordinates

$$U\big(\hat{x},F(r),q(t)\big)=A\big(q(t)\big)U_0\big(\hat{x},F(r)\big)A^\dagger\big(q(t)\big),\qquad A\big(q(t)\big)\in \mathrm{SU(3)}. \tag{II.3.1}$$

Because of the form of the ansatz U_0 (II.2.6), the unitary field U is invariant under the right U(1) transformation of the $A\big(q(t)\big)=D^{(\lambda,\mu)}(q(t))$ matrix, defined as

$$A\big(q(t)\big)\to A\big(q(t)\big)\exp\beta J^{(1,1)}_{(0,0,0)}\,. \tag{II.3.2}$$

Thus the seven-dimensional homogeneous space SU(3)/U(1), which is specified by seven real, independent parameters $q^k(t)$, has to be considered. The mathematical structure of the Skyrme model and its quantization problems on the coset space SU(3)/U(1) have been examined by several authors [35,75,104,105]. The canonical quantization procedure for the SU(3) Skyrme model in the fundamental representation has been considered by Fujii *et al.* [99]. Here the attention is on the representation dependence of the model. The Lagrangian (I.3.21) is considered quantum mechanically *ab initio*. The generalized coordinates $q^k(t)$ and velocities $(\mathrm{d}/\mathrm{d}t)q^k(t)=\dot{q}^k(t)$ satisfy the commutation relations

$$\big[\dot{q}^k,q^l\big]=-\mathrm{i}g^{kl}(q)\,, \tag{II.3.3}$$

where $g^{kl}(q)$ are functions of q^k only. Their form will be determined below. The commutation relation between a velocity component $\dot{q}^k$ and an arbitrary function of q is given by (I.7.3). For the time derivative the usual Weyl ordering (I.7.4) is adopted and the operator ordering is fixed by the form of the Lagrangian (I.3.21).

For the purpose of defining the metric tensor in the Lagrangian we use approximate expressions

$$\dot{A}A^\dagger\approx\frac{1}{2}\Big\{\dot{q}^i,C_i^{(Z,I,M)}(q)\Big\}\Big\langle\;\Big|\,J^{(1,1)}_{(Z,I,M)}\Big|\;\Big\rangle\,, \tag{II.3.4a}$$

$$A^\dagger\dot{A}\approx\frac{1}{2}\Big\{\dot{q}^i,C_i'^{(Z,I,M)}(q)\Big\}\Big\langle\;\Big|\,J^{(1,1)}_{(Z,I,M)}\Big|\;\Big\rangle\,. \tag{II.3.4b}$$

The ansatz (II.3.1) is then substituted in the Skyrme Lagrangian (I.3.21) following by an integration over the spatial coordinates. The Lagrangian is then obtained in terms of the collective coordinates and velocities. For the derivation of the canonical momenta it is sufficient to restrict the consideration to terms of the second order in the velocities (the terms of the first order vanish). This leads to

$$\begin{aligned}
L_{\text{Sk}} \approx & -\int \mathrm{d}r r^2 \Bigg\{ \sum_M (-1)^M \left\{ \dot{q}^i, C_i^{\prime(0,1,M)}(q) \right\} \left\{ \dot{q}^{i'}, C_{i'}^{\prime(0,1,-M)}(q) \right\} \\
& \times \frac{\pi}{3} \sin^2 F \left(f_\pi^2 + \frac{1}{e^2} \left(F'^2 + \frac{1}{r^2} \sin^2 F \right) \right) \\
& + \sum_{Z,M} (-1)^{Z+M} \left\{ \dot{q}^i, C_i^{\prime(Z,\frac{1}{2},M)}(q) \right\} \left\{ \dot{q}^{i'}, C_{i'}^{\prime(-Z,\frac{1}{2},-M)}(q) \right\} \\
& \times \frac{\pi}{4} (1 - \cos F) \left(f_\pi^2 + \frac{1}{4e^2} \left(F'^2 + \frac{2}{r^2} \sin^2 F \right) \right) \Bigg\} \\
\approx & \; \frac{1}{2} \dot{q}^\alpha g_{\alpha\beta}(q, F) \dot{q}^\beta + \left[(\dot{q})^0\text{– order terms} \right] \\
\approx & \; \frac{1}{8} \left\{ \dot{q}^\alpha, C_\alpha^{\prime(A)}(q) \right\} E_{(A)(B)}(F) \left\{ \dot{q}^\beta, C_\beta^{\prime(B)}(q) \right\} + \left[(\dot{q})^0\text{– order terms} \right],
\end{aligned} \tag{II.3.5}$$

the derivation of which is shown in (B.21–B.23).

The Lagrangian (II.3.5) is normalized by the factor (II.2.5). The metric tensor takes the form

$$g_{\alpha\beta}(q, F) = C_\alpha^{\prime(A)}(q) E_{(A)(B)}(F) C_\beta^{\prime(B)}(q), \tag{II.3.6}$$

where

$$E_{(Z,I,M)(Z',I',M')}(F) = -(-1)^{Z+M} a_I(F) \delta_{Z,-Z'} \delta_{I,I'} \delta_{M,-M'} \,. \tag{II.3.7}$$

Here the soliton moments of inertia are given as integrals over the dimensionless variable $\tilde{r} = e f_\pi r$:

$$a_0(F) = 0, \tag{II.3.8a}$$

$$a_{\frac{1}{2}}(F) = \frac{1}{e^3 f_\pi} \tilde{a}_{\frac{1}{2}}(F) = \frac{1}{e^3 f_\pi} 2\pi \int \mathrm{d}\tilde{r} \tilde{r}^2 \, (1 - \cos F) \left(1 + \frac{1}{4} F'^2 + \frac{1}{2\tilde{r}^2} \sin^2 F \right), \tag{II.3.8b}$$

$$a_1(F) = \frac{1}{e^3 f_\pi} \tilde{a}_1(F) = \frac{1}{e^3 f_\pi} \frac{8\pi}{3} \int \mathrm{d}\tilde{r} \tilde{r}^2 \sin^2 F \left(1 + F'^2 + \frac{1}{r^2} \sin^2 F \right). \tag{II.3.8c}$$

The canonical momentum, which is conjugate to q^β, is defined as

$$p_\beta^{(0)} = \frac{\partial L_{\text{Sk}}}{\partial \dot{q}^\beta} = \frac{1}{2} \left\{ \dot{q}^\alpha, g_{\alpha\beta} \right\} . \tag{II.3.9}$$

The canonical commutation relations

$$\left[q^\alpha, q^\beta \right] = \left[p_\alpha^{(0)}, p_\beta^{(0)} \right] = 0, \tag{II.3.10a}$$

$$\left[p_\beta^{(0)}, q^\alpha \right] = -\, \mathrm{i} \delta_{\alpha\beta} \,, \tag{II.3.10b}$$

then yield the following explicit form for the functions $g^{\alpha\beta}(q)$:

$$g^{\alpha\beta}(q) = (g_{\alpha\beta})^{-1} = C'^{\alpha}_{(\bar{A})}(q) E^{(\bar{A})(\bar{B})}(F) C'^{\beta}_{(\bar{B})}(q)\,, \tag{II.3.11}$$

where

$$E^{(\overline{Z,I,M})(\overline{Z',I',M'})}(F) = -(-1)^{Z+M} \frac{1}{a_I(F)} \delta_{Z,-Z'} \delta_{I,I'} \delta_{M,-M'}\,. \tag{II.3.12}$$

Note that here $E^{(0)(0)}(F)$ is left undefined. The summation over the indices $(\bar{A})$ denotes summation over the basis states (Z, I, M) of the irrep $(1, 1)$, excluding the state $(0, 0, 0)$. It proves convenient to introduce the reciprocal function matrix $C'^{\alpha}_{(\bar{A})}(q)$, the properties of which are described in Appendix B. The commutation relations of the momenta (II.3.10b) ensure the choice of the parameters q^{α} on the manifold SU(3)/U(1) (see [9, 93]). There is no need for an explicit parameterization of q^{α} at this moment.

Having determined the function $g^{\alpha\beta}(q)$ the following explicit expression of $A^{\dagger}\dot{A}$ is obtained:

$$\begin{aligned}
A^{\dagger}\dot{A} &= \frac{1}{2} D^{(\lambda,\mu)}(-q) \Big\{ \dot{q}^{\alpha}, \partial_{\alpha} D^{(\lambda,\mu)}(q) \Big\} \\
&= \frac{1}{2} \Big\{ \dot{q}^{\alpha}, C'^{(A)}_{\alpha}(q) \Big\} \Big\langle \; \Big| J^{(1,1)}_{(A)} \Big| \; \Big\rangle \\
&\quad - \frac{1}{2} \mathrm{i} C'^{\alpha}_{(\bar{A})}(q) E^{(\bar{A})(\bar{B})}(F) C'^{\beta}_{(\bar{B})}(q) C'^{(K)}_{\beta}(q) C'^{(K')}_{\alpha}(q) \Big\langle \; \Big| J^{(1,1)}_{(K)} J^{(1,1)}_{(K')} \Big| \; \Big\rangle \\
&= \frac{1}{2} \Big\{ \dot{q}^{\alpha}, C'^{(A)}_{\alpha}(q) \Big\} \Big\langle \; \Big| J^{(1,1)}_{(A)} \Big| \; \Big\rangle \\
&\quad - \frac{1}{2} \mathrm{i} E^{(\bar{A})(\bar{B})}(F) C'^{\beta}_{(\bar{B})}(q) C'^{(0)}_{\beta}(q) \Big(\Big\langle \; \Big| J^{(1,1)}_{(0)} J^{(1,1)}_{(\bar{A})} \Big| \; \Big\rangle + \Big\langle \; \Big| J^{(1,1)}_{(\bar{A})} J^{(1,1)}_{(0)} \Big| \; \Big\rangle \Big) \\
&\quad - \frac{3}{8} \mathrm{i} E^{(\bar{A})(\bar{B})}(F) C'^{\alpha}_{(\bar{A})}(q) C'^{(0)}_{\alpha}(q) C'^{\beta}_{(\bar{B})}(q) C'^{(0)}_{\beta}(q) \sum_{z,j}^{(\lambda,\mu)} \oplus\, y^2 \cdot \mathbb{1}_{z,j} \\
&\quad + \frac{\mathrm{i}}{2a_{\frac{1}{2}}(F)} C_2^{\mathrm{SU(3)}}(\lambda,\mu) \cdot \mathbb{1}_{\lambda,\mu} + \mathrm{i} \sum_{z,j}^{(\lambda,\mu)} \oplus \left(\frac{C^{\mathrm{SU(2)}}(j)}{2a_1(F)} - \frac{C^{\mathrm{SU(2)}}(j) + \frac{3}{4} y^2}{2a_{\frac{1}{2}}(F)} \right) \cdot \mathbb{1}_{z,j}.
\end{aligned} \tag{II.3.13}$$

Here $\mathbb{1}_{\lambda,\mu}$ is the unit matrix in the (λ, μ) irrep of SU(3) and $\mathbb{1}_{z,j}$ are unit matrices in the SU(2) irreps. Note that the inverse of the rotation represented by $D^{(\lambda,\mu)}(q)$ is denoted as $D^{(\lambda,\mu)}(-q)$.

The field expression (II.3.1) is substituted in the Lagrangian density (I.3.21) in order to obtain the explicit expression in terms of the collective coordinates and space coordinates. The explicit expression of $A^{\dagger}\dot{A}$ (II.3.13) and expressions with the SU(3) group generators (B.24–B.30) are employed as well. After a lengthy manipulation the complete expression of the Skyrme model Lagrangian density is obtained:

$$\begin{aligned}
\mathcal{L}_{\mathrm{Sk}} &= \frac{1}{4} \dim(\lambda,\mu) C_2^{\mathrm{SU(3)}}(\lambda,\mu) \\
&\quad \times \Bigg\{ -\frac{(1-\cos F)}{16} \left[f_{\pi}^2 + \frac{1}{4e^2} \left(F'^2 + \frac{2}{r^2} \sin^2 F \right) \right] \\
&\qquad\qquad \times \sum_{Z,M} (-1)^{Z+M} \Big\{ \dot{q}^{\alpha}, C'^{(Z,\frac{1}{2},M)}_{\alpha}(q) \Big\} \Big\{ \dot{q}^{\beta}, C'^{(Z,\frac{1}{2},M)}_{\beta}(q) \Big\} -
\end{aligned}$$

$$
\begin{aligned}
&-\frac{\sin^2 F}{8}\left[f_\pi^2+\frac{1}{e^2}\left(F'^2+\frac{1}{r^2}\sin^2 F\right)\right]\\
&\qquad\times\sum_{Z,M}\Big[(-1)^M\Big\{\dot q^{\alpha},C'^{(0,1,M)}_{\alpha}(q)\Big\}\Big\{\dot q^{\alpha'},C'^{(0,1,M)}_{\alpha'}(q)\Big\}\\
&\qquad\qquad-\left(\Big\{\dot q^{\alpha},C'^{(0,1,\cdot)}_{\alpha}(q)\Big\}\cdot\hat x\right)\left(\Big\{\dot q^{\alpha'},C'^{(0,1,\cdot)}_{\alpha'}(q)\Big\}\cdot\hat x\right)\Big]\\
&-\mathcal{M}_{\rm cl}-\Delta\mathcal{M}_1-\Delta\mathcal{M}_2-\Delta\mathcal{M}_3-\Delta\mathcal{M}'(q)\Bigg\}.
\end{aligned}
\tag{II.3.14}
$$

Here the following notation has been introduced:

$$
\begin{aligned}
\Delta\mathcal{M}_1(F)=&-\frac{\sin^2 F}{30a_1^2(F)}\Bigg\{f_\pi^2\left(12\sin^2 F\cdot C_2^{\rm SU(3)}(\lambda,\mu)-16\sin^2 F+15\right)\\
&+\frac{1}{2e^2}\Bigg(2F'^2\left(12\cos^2 F\cdot C_2^{\rm SU(3)}(\lambda,\mu)+16\sin^2 F-1\right)\\
&\qquad+\frac{\sin^2 F}{r^2}\left(6C_2^{\rm SU(3)}(\lambda,\mu)+7\right)\Bigg)\Bigg\};
\end{aligned}
\tag{II.3.15a}
$$

$$
\begin{aligned}
\Delta\mathcal{M}_2(F)=&-\frac{(1-\cos F)}{20a_{\frac12}^2(F)}\Bigg\{f_\pi^2\left(6\,(1-\cos F)\cdot C_2^{\rm SU(3)}(\lambda,\mu)+3\cos F+2\right)\\
&+\frac{1}{4e^2}\left(F'^2\left(6\,(1+\cos F)\cdot C_2^{\rm SU(3)}(\lambda,\mu)-3\cos F+2\right)+10\frac{\sin^2 F}{r^2}\right)\Bigg\};
\end{aligned}
\tag{II.3.15b}
$$

$$
\begin{aligned}
\Delta\mathcal{M}_3(F)=&-\frac{\sin^2 F}{30a_1(F)\,a_{\frac12}(F)}\Bigg\{f_\pi^2\left(12\,(1-\cos F)\cdot C_2^{\rm SU(3)}(\lambda,\mu)+16\cos F-1\right)\\
&+\frac{1}{2e^2}\left(F'^2\left(4\cos F\cdot\left(3C_2^{\rm SU(3)}(\lambda,\mu)-4\right)+15\right)+15\frac{\sin^2 F}{r^2}\right)\Bigg\};
\end{aligned}
\tag{II.3.15c}
$$

$$
\begin{aligned}
\Delta\mathcal{M}'(F,q)=&-\frac{3(1-\cos F)}{16a_{\frac12}^2(F)}\left\{f_\pi^2+\frac{1}{4e^2}\left(F'^2+\frac{2}{r^2}\sin^2 F\right)\right\}\\
&\times\left((-1)^{\bar{\bar A}}C'^{\alpha}_{(\bar{\bar A})}(q)C'^{(0)}_{\alpha}(q)C'^{\beta}_{(-\bar{\bar A})}(q)C'^{(0)}_{\beta}(q)+4\right).
\end{aligned}
\tag{II.3.15d}
$$

[The "4" in the last bracket on the last row is missing in the corresponding expression in ref. [99], the consequence of which is the appearance of a spurious term $-3/8a_{1/2}(F)$ in eq. (69b) of that paper (there are some minor misprints in that equation as well)]. The notation $(\bar{\bar A})$ indicates that only the states for which $I=\frac12$ and $Z=\pm\frac12$ are included. The $\Delta\mathcal{M}_k(F)$ terms may be interpreted as quantum mass corrections to the Lagrangian density. The $\Delta\mathcal{M}'(F,q)$ term depends on the quantum variables q^i and is an operator on the configuration space.

The integration of (II.3.14) over the space variables and normalization by the factor of (II.2.5) gives the Lagrangian

$$
\begin{aligned}
L_{\mathrm{Sk}} =& \int \mathcal{L}_{\mathrm{Sk}} \mathrm{d}^3x = \frac{1}{8}\left\{\dot{q}^i, C_i^{\prime(\bar{A})}(q)\right\} E_{(\bar{A})(\bar{B})}\left\{\dot{q}^{i'}, C_{i'}^{\prime(\bar{B})}(q)\right\} \\
&- M_{\mathrm{cl}} - \Delta M_1 - \Delta M_2 - \Delta M_3 - \Delta M'(q) \\
=& -\frac{1}{8a_{\frac{1}{2}}(F)}(-1)^{\bar{A}}\left\{\dot{q}^i, C_i^{\prime(\bar{A})}(q)\right\}\left\{\dot{q}^{i'}, C_{i'}^{\prime(-\bar{A})}(q)\right\} \\
&- \frac{1}{8}\left(\frac{1}{a_1(F)} - \frac{1}{a_{\frac{1}{2}}(F)}\right)(-1)^M\left\{\dot{q}^i, C_i^{\prime(0,1,M)}(q)\right\}\left\{\dot{q}^{i'}, C_{i'}^{\prime(0,1,-M)}(q)\right\} \\
&- M_{\mathrm{cl}} - \Delta M_1 - \Delta M_2 - \Delta M_3 - \Delta M'(q).
\end{aligned}
\tag{II.3.16}
$$

Here $M_{\mathrm{cl}} = \frac{f_\pi}{e}\tilde{M}_{\mathrm{cl}} = \int \mathrm{d}^3x \mathcal{M}_{\mathrm{cl}}(F)$, $\Delta M_k = e^3 f_\pi \Delta\tilde{M}_k = \int \mathrm{d}^3x \Delta\mathcal{M}_k(F)$, $\Delta M'(q) = \int \mathrm{d}^3x \Delta\mathcal{M}'(q)$. The tilde placed over a letters mark integration over the dimensionless variable.

4. Structure of the Lagrangian and the Hamiltonian

The standard form of the Wess-Zumino action (I.6.18) is updated by a normalization factor N':

$$
S_{\mathrm{WZ}}\big(U(\mathbf{x})\big) = -\frac{\mathrm{i}N_{\mathrm{c}}}{240\pi^2 N'}\int_{\mathrm{D}_5} \mathrm{d}^5x\, \epsilon^{\mu\nu\lambda\rho\sigma}\,\mathrm{Tr}\Big(R_\mu R_\nu R_\lambda R_\rho R_\sigma\Big). \tag{II.4.1}
$$

The derivation of the contribution of the Wess-Zumino term to the effective Lagrangian in the framework of the collective coordinate formalism is given in Ref. [106]. Application of Stoke's theorem leads to the following form

$$
\begin{aligned}
L_{\mathrm{WZ}}(q,\dot{q}) =& -\frac{\mathrm{i}N_{\mathrm{c}}}{24\pi^2 N'}\int_{\mathrm{D}_3} \mathrm{d}^3x\, \epsilon^{mjk}\,\mathrm{Tr}\left(\left(\partial_m U_0\right) U_0^\dagger \left(\partial_j U_0\right) U_0^\dagger \left(\partial_k U_0\right) U_0^\dagger J^{(1,1)}_{(0,0,0)}\right) \\
&\times \frac{1}{2}\Big\{\dot{q}^\alpha, C_\alpha^{\prime(0)}(q)\Big\} \\
=& -\frac{\mathrm{i}N_{\mathrm{c}}}{4\sqrt{3}\pi^2 N'}\int_{\mathrm{D}_3} \mathrm{d}^3x \frac{\sin^2 F(r)}{r^2} F'(r) \sum_{z,j}^{(\lambda,\mu)} y j(j+1)(2j+1)\Big\{\dot{q}^\alpha, C_\alpha^{\prime(0)}(q)\Big\} \\
=& -\lambda' \frac{\mathrm{i}}{2}\Big\{\dot{q}^\alpha, C_\alpha^{\prime(0)}(q)\Big\}.
\end{aligned}
\tag{II.4.2}
$$

Here

$$
\lambda' = \frac{\sqrt{3}N_{\mathrm{c}}B}{40N'}\,\dim(\lambda,\mu) C_3^{\mathrm{SU(3)}}(\lambda,\mu)\,. \tag{II.4.3}
$$

The coefficient λ' depends on the representation (λ,μ) . Following Witten [3] the normalization factor is chosen to be $N' = \dim(\lambda,\mu) C_3^{\mathrm{SU(3)}}(\lambda,\mu)/20$ so that $\lambda' = N_{\mathrm{c}}B/2\sqrt{3}$. In the case of the fundamental representation $N' = 1$. Here the coefficient λ' only constrains the states of the system. Since the cubic Casimir operator $\hat{C}_3^{\mathrm{SU(3)}}$ (II.2.12) vanishes in the self-adjoint representations $\lambda = \mu$, the WZ term (II.4.2) also vanishes in those representations.

The Lagrangian of the system with the inclusion of the WZ term is

$$
L' = L_{\mathrm{Sk}} + L_{\mathrm{WZ}}\,. \tag{II.4.4}
$$

There are seven collective coordinates. The momenta p_α, that are canonically conjugate to q^α, are defined as

$$p_\alpha = \frac{\partial L'}{\partial \dot{q}^\alpha} = \frac{1}{2}\left\{\dot{q}^\beta, g_{\beta\alpha}\right\} - \mathrm{i}\lambda' C'^{(0)}_\alpha(q)\,. \tag{II.4.5}$$

These satisfy the canonical commutation relations (II.3.10b). The WZ term may be considered as an external potential in the system [35, 104, 105]. Seven right transformation generators may be defined as

$$\hat{R}_{(\bar{A})} = \frac{\mathrm{i}}{2}\left\{p_\alpha + \mathrm{i}\lambda' C'^{(0)}_\alpha(q), C'^\alpha_{(\bar{A})}(q)\right\} = \frac{\mathrm{i}}{2}\left\{\dot{q}^\beta, C'^{(\bar{B})}_\beta(q)\right\} E_{(\bar{B})(\bar{A})}. \tag{II.4.6}$$

They satisfy the following commutation relations

$$\begin{aligned}\left[\hat{R}_{(\bar{A}')}, \hat{R}_{(\bar{A}'')}\right] &= -\sqrt{3}\begin{bmatrix}(1,1) & (1,1) & (1,1)_a \\ (\bar{A}') & (\bar{A}'') & (\bar{A})\end{bmatrix}\hat{R}_{(\bar{A})} \\ &\quad + \sqrt{3}z''\left\{C'^\alpha_{(\bar{A}')}(q)C'^{(0)}_\alpha(q), \hat{R}_{(\bar{A}'')}\right\} \\ &\quad - \sqrt{3}z'\left\{C'^\alpha_{(\bar{A}'')}(q)C'^{(0)}_\alpha(q), \hat{R}_{(\bar{A}')}\right\},\end{aligned} \tag{II.4.7}$$

and the right transformation rules for the irrep matrices are

$$\begin{aligned}\left[\hat{R}_{(\bar{K})}, D^{(\lambda,\mu)}_{(A)(A')}(q)\right] &= D^{(\lambda,\mu)}_{(A)(A'')}(q)\left\langle \begin{matrix}(\lambda,\mu)\\ A''\end{matrix}\right| J^{(1,1)}_{(\bar{K})}\left|\begin{matrix}(\lambda,\mu)\\ A'\end{matrix}\right\rangle \\ &\quad - \frac{\sqrt{3}}{2}y' C'^\alpha_{(\bar{K})}(q)C'^{(0)}_\alpha(q)D^{(\lambda,\mu)}_{(A)(A')}(q)\,.\end{aligned} \tag{II.4.8}$$

It is convenient to define the eighth transformation generator formally as [99]

$$\hat{R}_{(0)} = -\lambda'\,. \tag{II.4.9}$$

The SU(2) subalgebra of the generators $\hat{R}_{(0,1,M)}$ satisfies the standard SU(2) commutation relations. These may be interpreted as spin operators because their acting on the unitary field (II.3.1) can be realized as a spatial rotation of the skyrmion only

$$\left[\hat{R}_{(0,1,M)}, A(q)U_0(x)A^\dagger(q)\right] = A(q)\left[J^{(1,1)}_{(0,1,M)}, U_0(x)\right]A^\dagger(q). \tag{II.4.10}$$

Eight left transformation generators are defined as

$$\begin{aligned}\hat{L}_{(B)} &= \frac{1}{2}\left\{\hat{R}_{(A)}, D^{(1,1)}_{(A)(B)}(-q)\right\} \\ &= \frac{\mathrm{i}}{2}\left\{p_\beta + \mathrm{i}\lambda' C'^{(0)}_\beta(q), K^\beta_{(B)}(q)\right\} + \lambda' D^{(1,1)}_{(0)(B)}(-q),\end{aligned} \tag{II.4.11}$$

where

$$K^\beta_{(B)}(q) = C'^\beta_{(\bar{A})}(q)D^{(1,1)}_{(\bar{A})(B)}(-q)\,, \tag{II.4.12}$$

the properties of which follow from (B.3)

$$K^{\beta''}_{(B'')}(q)\partial_{\beta''}K^{\beta'}_{(B')}(q) - K^{\beta''}_{(B')}(q)\partial_{\beta''}K^{\beta'}_{(B'')}(q) = \sqrt{3}\begin{bmatrix}(1,1) & (1,1) & (1,1)_a \\ (B'') & (B') & (B)\end{bmatrix}K^{\beta'}_{(B)}(q). \tag{II.4.13}$$

By making use of (II.4.7) it may be proven that

$$\left[\hat{L}_{(B')}, \hat{L}_{(B'')}\right] = \sqrt{3} \begin{bmatrix} {}^{(1,1)}_{(B')} & {}^{(1,1)}_{(B'')} & {}^{(1,1)_a}_{(B)} \end{bmatrix} \hat{L}_{(B)} \,. \tag{II.4.14}$$

Three right transformation generators or spin operators $\hat{R}_{(0,1,M)}$ commute with the left transformation generators

$$\left[\hat{R}_{(0,1,M)}, \hat{L}_{(B)}\right] = 0 \,. \tag{II.4.15}$$

The left transformation rules for the irrep matrices are

$$\begin{aligned}\left[\hat{L}_{(B)}, D^{(\lambda,\mu)}_{(A')(A)}(q)\right] = {} & \left\langle A' \,\middle|\, J^{(1,1)}_{(B)} \,\middle|\, A'' \right\rangle D^{(\lambda,\mu)}_{(A'')(A)}(q) - \frac{\sqrt{3}}{2} y_A \, D^{(1,1)}_{(0)(B)}(-q) D^{(\lambda,\mu)}_{(A')(A)}(q) \\ & - \frac{\sqrt{3}}{2} y_A \, C'^{(0)}_{\alpha}(q) C'^{\alpha}_{(\bar{B}')}(q) D^{(1,1)}_{(\bar{B}')(B)}(-q) D^{(\lambda,\mu)}_{(A')(A)}(q).\end{aligned} \tag{II.4.16}$$

It is straightforward to derive the following result

$$\begin{aligned}(-1)^B \hat{L}_{(B)} \hat{L}_{(-B)} = {} & (-1)^{\bar{A}} \hat{R}_{(\bar{A})} \hat{R}_{(-\bar{A})} + \lambda'^2 - \frac{3}{4} \\ & - \frac{3}{16} (-1)^{\bar{\bar{A}}} C'^{(0)}_{\alpha}(q) C'^{\alpha}_{(\bar{\bar{A}})}(q) C'^{(0)}_{\beta}(q) C'^{\beta}_{(-\bar{\bar{A}})}(q) \,.\end{aligned} \tag{II.4.17}$$

The effective Lagrangian, which includes the WZ term takes the form:

$$\begin{aligned}L_{\text{eff}} = {} & \frac{1}{2a_{\frac{1}{2}}(F)} (-1)^{\bar{A}} \hat{R}_{(\bar{A})} \hat{R}_{(-\bar{A})} + \left(\frac{1}{2a_1(F)} - \frac{1}{2a_{\frac{1}{2}}(F)} \right) \left(\hat{R}_{(0,1,\cdot)} \cdot \hat{R}_{(0,1,\cdot)} \right) \\ & - \lambda' \frac{\mathrm{i}}{2} \left\{ \dot{q}^{\alpha}, C'^{(0)}_{\alpha}(q) \right\} - M_{\text{cl}} - \Delta M_1 - \Delta M_2 - \Delta M_3 - \Delta M'(q) \\ = {} & \frac{1}{2a_{\frac{1}{2}}(F)} \left((-1)^A \hat{L}_{(A)} \hat{L}_{(-A)} - \lambda'^2 \right) + \left(\frac{1}{2a_1(F)} - \frac{1}{2a_{\frac{1}{2}}(F)} \right) \left(\hat{R}_{(0,1,\cdot)} \cdot \hat{R}_{(0,1,\cdot)} \right) \\ & - \lambda' \frac{\mathrm{i}}{2} \left\{ \dot{q}^{\alpha}, C'^{(0)}_{\alpha}(q) \right\} - M_{\text{cl}} - \Delta M_1 - \Delta M_2 - \Delta M_3.\end{aligned} \tag{II.4.18}$$

Note that the $\Delta M'(q)$ term which depends on the quantum variables vanishes due to the introduction of the left translation generators in the Lagrangian expression (II.4.18).

For the purpose of obtaining the Euler-Lagrange equations that are consistent with the canonical equation of motion of the Hamiltonian, the general method of quantization on a curved space is employed. Sugano *et al.* [10, 90–92] suggested to introduce the following auxiliary function

$$\begin{aligned}Z(q) = {} & -\frac{1}{16} f^{ab} f^{cd} f^{ek} \left(\partial_a g_{cd}\right) \left(\partial_b g_{ek}\right) - \frac{1}{4} \partial_a \left(f^{ab} f^{cd} \partial_b g_{cd} \right) - \frac{1}{4} \partial_a \partial_b f^{ab} \\ = {} & -\frac{1}{4} \partial_b C'^{a}_{(\bar{A})}(q) E^{(\bar{A})(\bar{B})} \partial_a C'^{b}_{(\bar{B})}(q) \\ & + \frac{3}{16 a_{\frac{1}{2}}(F)} \left((-1)^{\bar{\bar{A}}} C'^{(0)}_{a}(q) C'^{a}_{(\bar{\bar{A}})}(q) C'^{b}_{(-\bar{\bar{A}})}(q) C'^{(0)}_{b}(q) + 4 \right) .\end{aligned} \tag{II.4.19}$$

Using it, the covariant kinetic term may be defined as

$$2K = \frac{1}{2}\left\{p_\alpha + \mathrm{i}\lambda' C'^{(0)}_\alpha(q), \dot{q}^\alpha\right\} - Z(q)$$
$$= \frac{1}{a_{\frac{1}{2}}(F)}\left((-1)^A \hat{L}_{(A)}\hat{L}_{(-A)} - \lambda'^2\right) + \left(\frac{1}{a_1(F)} - \frac{1}{a_{\frac{1}{2}}(F)}\right)\left(\hat{R}_{(0,1,\cdot)} \cdot \hat{R}_{(0,1,\cdot)}\right). \quad \text{(II.4.20)}$$

According to the prescription [10,90–92] the effective Hamiltonian (with the constraint (II.4.9)) is constructed in the standard form as

$$H = \frac{1}{2}\{p_\alpha, \dot{q}^\alpha\} - L_{\text{eff}} - Z(q) = K + \Delta M_1 + \Delta M_2 + \Delta M_3 + M_{\text{cl}}\,. \quad \text{(II.4.21)}$$

Upon renormalization the Lagrangian density (II.3.14) may be reexpressed in terms of the left and right transformation generators. In turn the effective Hamiltonian density without the symmetry breaking term takes the form

$$\mathcal{H}_{\text{Sk}} = \frac{1-\cos F}{4a^2_{\frac{1}{2}}(F)}\left(f_\pi^2 + \frac{1}{4e^2}\left(F'^2 + \frac{2}{r^2}\sin^2 F\right)\right)$$
$$\times\left((-1)^A\hat{L}_{(A)}\hat{L}_{(-A)} - \left(\hat{R}_{(0,1,\cdot)}\cdot\hat{R}_{(0,1,\cdot)}\right) - \lambda'^2\right)$$
$$+ \frac{\sin^2 F}{2a_1^2(F)}\left(f_\pi^2 + \frac{1}{e^2}\left(F'^2 + \frac{1}{r^2}\sin^2 F\right)\right)$$
$$\times\left(\left(\hat{R}_{(0,1,\cdot)}\cdot\hat{R}_{(0,1,\cdot)}\right) - \left(\hat{R}_{(0,1,\cdot)}\cdot\hat{x}\right)\left(\hat{R}_{(0,1,\cdot)}\cdot\hat{x}\right)\right)$$
$$+ \Delta\mathcal{M}_1 + \Delta\mathcal{M}_2 + \Delta\mathcal{M}_3 + \mathcal{M}_{\text{cl}}\,. \quad \text{(II.4.22)}$$

The products of the spin operators $\hat{R}_{(0,1,M)}$ may be separated into the scalar and tensorial terms as

$$\left(\hat{R}_{(0,1,\cdot)}\cdot\hat{R}_{(0,1,\cdot)}\right) - \left(\hat{R}_{(0,1,\cdot)}\cdot\hat{x}\right)\left(\hat{R}_{(0,1,\cdot)}\cdot\hat{x}\right) =$$
$$= \frac{2}{3}\left(\hat{R}_{(0,1,\cdot)}\cdot\hat{R}_{(0,1,\cdot)}\right) - \frac{4\pi}{3}Y^*_{2,M+M'}(\vartheta,\varphi)\begin{bmatrix} 1 & 1 & 2 \\ M & M' & M+M' \end{bmatrix}\hat{R}_{(0,1,M)}\hat{R}_{(0,1,M')}. \quad \text{(II.4.23)}$$

The covariant kinetic term (II.4.20) is a differential operator constructed from the SU(3) left and the SU(2) right transformation generators. The eigenstates of the Hamiltonian (II.4.21) are

$$\left|\begin{matrix}(\Lambda, M)\\ Y, T, M_T; Y', S, M_S\end{matrix}\right\rangle = \sqrt{\dim(\Lambda, M)}\,D^{*(\Lambda,M)}_{(Y,T,M_T)(Y',S,M_S)}(q)\left|0\right\rangle\,. \quad \text{(II.4.24)}$$

Here the quantities D on the right-hand side are the complex conjugate Wigner matrix elements of the (Λ, M) irrep of SU(3) in terms of the quantum variables q^k. The topology of the eigenstates can be nontrivial and the quantum states contain the eighth “unphysical” quantum variable q^0.

Due to tensorial part (II.4.23) the matrix elements of the Hamiltonian density (II.4.22) for states with the spin $S > \frac{1}{2}$ have quadrupole moments. In the case of $S = \frac{1}{2}$ the matrix element of the second rank operator on the right hand side of (II.4.23) vanishes.

5. The symmetry breaking mass term

The chiral symmetry breaking mass term for the SU(3) soliton was defined in (II.2.9). With the ansatz (II.3.1) in (II.2.9), the symmetry breaking density operator for the general irrep (λ, μ) becomes

$$\mathcal{L}_{\text{SB}} = -\mathcal{M}_{\text{SB}} = -\frac{1}{N}\frac{f_\pi^2}{4}\Bigg(m_0^2 \operatorname{Tr}\left(U_0{+}U_0^\dagger - 2\cdot\mathbb{1}\right) - 2m_8^2 \operatorname{Tr}\left(\left(U_0{+}U_0^\dagger\right) J^{(1,1)}_{(0,0,0)}\right) D^{(1,1)}_{(0)(0)}(-q)\Bigg). \tag{II.5.1}$$

The operator (II.5.1) contains the matrix elements $D^{(1,1)}$, which depend on the quantum variables q^α. In this form the $\mathcal{L}_{\text{SB}}$ operator mixes the representations (Λ, M) of the eigenstates of the Hamiltonian [107]. Therefore the physical states of the system have to be calculated by a diagonalisation of the Hamiltonian. Since the mass term is a minor part of the Lagrangian it may considered as a perturbation in the SU(3) representation (Λ, M).

For a given irrep (λ, μ), using symbolical summation (see Appendix D), the symmetry breaking term depends on the profile function $F(r)$ as

$$\begin{aligned}\operatorname{Tr}\left(U_0{+}U_0^\dagger - 2\cdot\mathbb{1}\right) &= 2\sum_{z,j}^{(\lambda,\mu)}\left(\sum_{m=-j}^{j}\cos 2mF(r)\right) - 2\dim(\lambda,\mu)\\ &= 2\,\frac{\sin(1+\lambda)F(r)+\sin(1+\mu)F(r)-\sin(\lambda+\mu+2)F(r)}{2\sin F(r)-\sin 2F(r)}\\ &\quad - 2\dim(\lambda,\mu)\,.\end{aligned} \tag{II.5.2}$$

Further development of the expression (II.5.1) leads to:

$$\begin{aligned}&\operatorname{Tr}\left(\left(U_0{+}U_0^\dagger\right) J^{(1,1)}_{(0,0,0)}\right) = 2\sum_{z,j}^{(\lambda,\mu)} 2\sqrt{3}\left(\frac{1}{3}(\lambda-\mu)+z\right)\left(\sum_{m=-j}^{j}\cos 2mF(r)\right)\\ &= \frac{2\sqrt{3}}{2\sin F(r)-\sin 2F(r)}\\ &\times\Bigg\{\frac{1}{2}(1+\mu)\left(\sin(1+\mu)F(r)-\sin(\lambda+\mu+2)F(r)\right)\\ &+\frac{1}{3}(\lambda-\mu)\left(\sin(1+\lambda)F(r)+\sin(1+\mu)F(r)-\sin(\lambda+\mu+2)F(r)\right)\\ &+\frac{1}{2}(1+\lambda)\Big(\left(\sin F(r)-\sin(2+\mu)F(r)\right)\cos\lambda F(r)\\ &-\left(\cos F(r)-\cos(2+\mu)F(r)\right)\sin\lambda F(r)\Big)\Bigg\}.\end{aligned} \tag{II.5.3}$$

For high irrep (λ, μ) the dependence of the symmetry breaking term on the profile function $F(r)$ differs from that in the fundamental representation $(1, 0)$ significantly. In the latter representation the symmetry breaking term takes the standard form

$$\mathcal{M}_{\text{SB}} = f_\pi^2\,(1-\cos F)\left(m_0^2 + \frac{m_8^2}{\sqrt{3}}\,D^{(1,1)}_{(0)(0)}(-q)\right). \tag{II.5.4}$$

In the case of the representation $(2,0)$ the expression is:

$$\mathcal{M}_{\text{SB}} = \frac{f_\pi^2}{5}\left(m_0^2(1-\cos F + 2\sin^2 F) - \frac{m_8^2}{\sqrt{3}}(1-\cos F - 4\sin^2 F)D^{(1,1)}_{(0)(0)}(-q)\right). \quad \text{(II.5.5)}$$

Note that in both cases the asymptotical behavior of the symmetry breaking terms at large distance are different.

6. Summary

The SU(3) Skyrme model was quantized canonically in the framework of the collective coordinate formalism in the representations of an arbitrary dimension. This lead to the complete quantum mechanical structure of the model on the homogeneous space of SU(3)/U(1). The derivation extended previuos studies made in the fundamental representation for SU(2) and SU(3) [9,93,99] and those of general representations of SU(2) [11–13]. The explicit representation dependence of the quantum corrections to the Skyrme model Lagrangian was derived. This dependence is nontrivial, especially for the Wess-Zumino and the symmetry breaking terms. The operators that form the Hamiltonian were shown to have well defined group-theoretical properties.

In general the choice of the irrep that is used for the unitary field depends on the phenomenological aspects of the physical system to which the model is applied. Formally the variation of the irrep can be interpreted as a modification of the Skyrme model. The representation dependence of the Wess-Zumino term was shown to be absorbable into the normalization factor, with an exception of the self adjoint irreps in which this case term vanishes. The symmetry breaking term has different functional dependence on the profile function $F(r)$ in different irreps. In the case of the self adjoint representations the symmetry breaking term, which is proportional to the coefficient m_8^2 also vanishes.

The effective Hamiltonian (II.4.21) commutes with the left transformation generators $\hat{L}_{(A)}$ and the right transformation (spin) generators $\hat{R}_{(0,1,M)}$

$$\left[\hat{L}_{(A)}, H\right] = \left[\hat{R}_{(0,1,M)}, H\right] = 0\,, \quad \text{(II.6.1)}$$

which ensures that the states (II.4.24) are the eigenstates of the effective Hamiltonian.

The symmetry breaking term does, however, not commute with the left generators

$$\left[\hat{L}_{(Z,\frac{1}{2},M)}, M_{\text{SB}}\right] \neq 0\,, \quad \text{(II.6.2)}$$

and therefore this term mixes the states in different representations (Λ, M).

A new result of this investigation is the tensor term (II.4.23) in the Hamiltonian density operator (II.4.22). Because of the tensor operator the states with the spin $S > \frac{1}{2}$ have quadrupole moments.

Lets Consider the energy functional of the quantum skyrmion in the states of the (Λ, M) irrep. The problem is simplified if the symmetry breaking term that leads to the representation mixing is dropped

$$\begin{aligned} E(F) = {} & \frac{C_2^{\text{SU(3)}}(\Lambda, M) - \lambda'^2}{a_{\frac{1}{2}}(F)} + \left(\frac{1}{a_1(F)} - \frac{1}{a_{\frac{1}{2}}(F)}\right)S(S+1) \\ & + \Delta M_1 + \Delta M_2 + \Delta M_3 + M_{\text{cl}}. \end{aligned} \quad \text{(II.6.3)}$$

The variational condition for the energy is

$$\frac{\delta E(F)}{\delta F} = 0\,, \tag{II.6.4}$$

with the usual boundary conditions $F(0) = \pi, F(\infty) = 0$. At large distances this equation reduces to the asymptotic form

$$\tilde{r}^2 F'' + 2\tilde{r}F' - (2 + \tilde{m}^2\tilde{r}^2)F = 0, \tag{II.6.5}$$

where the quantity $\tilde{m}^2$ is defined as

$$\begin{aligned}\tilde{m}^2 = -e^4 \bigg(& \frac{1}{4\tilde{a}^2_{\frac{1}{2}}(F)} \big(C_2^{\mathrm{SU(3)}}(\Lambda, M) - S(S+1) - \lambda'^2 + 1\big) + \frac{2\,S(S+1)+3}{3\,\tilde{a}^2_1(F)} \\ & + \frac{8\Delta\tilde{M}_1 + 4\Delta\tilde{M}_3}{3\,\tilde{a}_1(F)} + \frac{\Delta\tilde{M}_3 + 2\Delta\tilde{M}_2}{2\,\tilde{a}_{\frac{1}{2}}(F)} + \frac{1}{\tilde{a}_1(F)\tilde{a}_{\frac{1}{2}}(F)}\bigg).\end{aligned} \tag{II.6.6}$$

The corresponding asymptotic solution takes the form

$$F(\tilde{r}) = k\left(\frac{\tilde{m}}{\tilde{r}} + \frac{1}{\tilde{r}^2}\right)\exp(-\tilde{m}\tilde{r}), \qquad k = \text{const.} \tag{II.6.7}$$

The quantum corrections depends on the irrep (λ, μ) to which the unitary field $U(\mathbf{x}, t)$ belongs as well as on the state irrep (Λ, M) and the spin S. This bears on the stability of the quantum skyrmion, which is stable if the integrals (II.3.8b, II.3.8c) and ΔM_k converge. This requirement is satisfied only if $\tilde{m}^2 > 0$, which is true only in the presence of the negative quantum mass corrections ΔM_k. It is the absence of this term that leads to the instability of the skyrmion in the semiclassical approach [13] in the SU(2) case. Note that in the quantum treatment the profile function $F(\tilde{r})$ has the asymptotic exponential behavior (II.6.7) even in the chiral limit.

III. Quantum SU(3) Skyrme model with the noncanonically embedded SO(3) soliton

The aim of this chapter is to discuss the group-theoretical aspects of the canonical quantization of the SU(3) Skyrme model with a new SO(3) ansatz which differs from the one proposed by A.P. Balachandran *et al.* [106]. The ansatz is defined in the noncanonical SU(3) $\supset$ SO(3) bases. These bases were developed by J.P. Elliott to consider the collective motion of nuclei [108]. The canonical quantization of the soliton leads to new expressions of the moments of inertia and the negative quantum corrections.

1. Definitions for a soliton in the noncanonical SU(3) bases

The unitary field $U(\mathbf{x},t)$ is defined for the SU(3) group in the arbitrary irrep (λ,μ). The modified Skyrme model is based on the standard Lagrangian density (I.3.21) where the "right" and "left" chiral currents are defined as

$$R_\mu = (\partial_\mu U)\,U^\dagger = \partial_\mu \alpha^i C_i^{(L,M)}(\alpha)\left\langle \;\left| J^{(1,1)}_{(L,M)} \right|\; \right\rangle, \tag{III.1.1a}$$

$$L_\mu = U^\dagger\,(\partial_\mu U) = \partial_\mu \alpha^i C_i^{\prime(L,M)}(\alpha)\left\langle \;\left| J^{(1,1)}_{(L,M)} \right|\; \right\rangle. \tag{III.1.1b}$$

They have values on the SU(3) algebra [14]. The explicit expressions of functions $C_i^{(L,M)}(\alpha)$ and $C_i^{\prime(L,M)}(\alpha)$ depend on fixing of eight parameters α^i of the group. $J^{(1,1)}_{(L,M)}$ are the generators of the group in the irrep (λ,μ). We will consider here the unitary field $U(\mathbf{x},t)$ in the fundamental representation $(1,0)$ based on the modified ansatz.

We suggest the generalization of the usual hedgehog ansatz for any irrep j of the SU(2) group [11]

$$\exp \mathrm{i}(\boldsymbol{\tau}\cdot\hat{x})F(r) \to \exp \mathrm{i}2(\hat{J}\cdot\hat{x})F(r) = U_0\big(\hat{x},F(r)\big) = D^j\big(\hat{x},F(r)\big), \tag{III.1.2}$$

here $\hat{J}$ is a generator of the SU(2) group in the irrep j. The particular Wigner D^j matrix elements, which represent the hedgehog ansatz for the irrep j, are

$$D^j_{a,a'}\big(\hat{x},F(r)\big) = \frac{2\sqrt{\pi}}{2j+1} w^j_l(F) \begin{bmatrix} j & l & j \\ a & m & a' \end{bmatrix} Y_{l,m}(\theta,\varphi). \tag{III.1.3}$$

The boundary conditions $F(0)=\pi$ and $F(\infty)=0$ ensure the winding (baryon) number $B=1$ for all irreps j due to the reason that the classical Lagrangian and the winding number have the same factor $N = \frac{2}{3}j(j+1)(2j+1)$ which can be reduced [13]. In this work we chose the ansatz in the three dimensional SU(2) group representation which is also the fundamental representation of the SO(3) group. The radial functions in (III.1.3) for this ansatz are as follows:

$$
\begin{aligned}
&w_0^1(F) = \sqrt{2}\left(3 - 4\sin^2 F\right), && w_0^2(F) = \left(5 - 20\sin^2 F + 16\sin^4 F\right),\\
&w_1^1(F) = \mathrm{i}2\sqrt{3}\sin 2F, && w_1^2(F) = \mathrm{i}\sqrt{2}\left(\sin 2F + 2\sin 4F\right),\\
&w_2^1(F) = -\,4\sin^2 F, && w_2^2(F) = -\,\frac{2\sqrt{2\cdot 5}}{\sqrt{7}}\left(7 - 8\sin^2 F\right)\sin^2 F,\\
& && w_3^2(F) = -\,\mathrm{i}\,4\sqrt{2}\sin^2 F\sin 2F,\\
& && w_4^2(F) = \frac{8\sqrt{2}}{\sqrt{7}}\sin^4 F.
\end{aligned}
\tag{III.1.4}
$$

It is convenient to define the noncanonical bases of the SU(3) irrep state vectors by a reduction chain on the subgroup SU(3) $\supset$ SO(3). As we shall see later the structure of the quantum skyrmion depends on a choice of the bases for the ansatz. For general SU(3) irreps (λ,μ) the SO(3) subgroup parameters (L,M) and their multiplicity can be sorted out by different methods, see [109, 110]. Here considered the fundamental $(1,0)$ and the adjoint $(1,1)$ representations of the SU(3) group are multiplicity free. The relations between the canonical basis vectors $\left|{(1,0) \atop z,j,m}\right\rangle$ where hypercharge $y = \frac{2}{3}(\mu-\lambda) - 2z$ (the reduction chain SU(3) $\supset$ SU(2)), and the noncanonical basis state vectors $\left|{(1,0) \atop L,M}\right\rangle$ are straightforward in the fundamental representation:

$$
\left|{(1,0) \atop \frac{1}{2},\frac{1}{2},\frac{1}{2}}\right\rangle = \left|{(1,0) \atop 1,1}\right\rangle; \quad \left|{(1,0) \atop 0,0,0}\right\rangle = \left|{(1,0) \atop 1,0}\right\rangle; \quad \left|{(1,0) \atop \frac{1}{2},\frac{1}{2},-\frac{1}{2}}\right\rangle = \left|{(1,0) \atop 1,-1}\right\rangle. \tag{III.1.5}
$$

The relations between the canonical basis vectors and the noncanonical basis state vectors of the adjoint $(1,1)$ representation are following:

$$
\begin{aligned}
&\left|{(1,1) \atop 0,1,1}\right\rangle = -\left|{(1,1) \atop 2,2}\right\rangle; \qquad \left|{(1,1) \atop 0,0,0}\right\rangle = -\left|{(1,1) \atop 2,0}\right\rangle;\\
&\left|{(1,1) \atop 0,1,0}\right\rangle = \left|{(1,1) \atop 1,0}\right\rangle; \qquad \left|{(1,1) \atop 0,1,-1}\right\rangle = \left|{(1,1) \atop 2,-2}\right\rangle;\\
&\left|{(1,1) \atop \frac{1}{2},\frac{1}{2},\frac{1}{2}}\right\rangle = -\frac{1}{\sqrt{2}}\left|{(1,1) \atop 2,1}\right\rangle + \left|{(1,1) \atop 1,1}\right\rangle;\\
&\left|{(1,1) \atop \frac{1}{2},\frac{1}{2},-\frac{1}{2}}\right\rangle = \frac{1}{\sqrt{2}}\left|{(1,1) \atop 2,-1}\right\rangle + \left|{(1,1) \atop 1,-1}\right\rangle;\\
&\left|{(1,1) \atop -\frac{1}{2},\frac{1}{2},-\frac{1}{2}}\right\rangle = -\frac{1}{\sqrt{2}}\left|{(1,1) \atop 2,-1}\right\rangle + \left|{(1,1) \atop 1,-1}\right\rangle;\\
&\left|{(1,1) \atop -\frac{1}{2},\frac{1}{2},\frac{1}{2}}\right\rangle = -\frac{1}{\sqrt{2}}\left|{(1,1) \atop 2,1}\right\rangle - \frac{1}{\sqrt{2}}\left|{(1,1) \atop 1,1}\right\rangle.
\end{aligned}
\tag{III.1.6}
$$

The system of the noncanonical SU(3) generator in terms of the canonical generators $J^{(1,1)}_{(Z,I,M)}$, which are defined in (II.1.3), can be expressed as follows:

$$\begin{aligned}
J_{(1,1)} &= \sqrt{2}\left(J^{(1,1)}_{(\frac{1}{2},\frac{1}{2},\frac{1}{2})} - J^{(1,1)}_{(-\frac{1}{2},\frac{1}{2},\frac{1}{2})}\right), \qquad & J_{(1,0)} &= 2J^{(1,1)}_{(0,1,0)}\,, \\
J_{(1,-1)} &= \sqrt{2}\left(J^{(1,1)}_{(-\frac{1}{2},\frac{1}{2},-\frac{1}{2})} + J^{(1,1)}_{(\frac{1}{2},\frac{1}{2},-\frac{1}{2})}\right), \qquad & J_{(2,2)} &= -2J^{(1,1)}_{(0,1,1)}\,, \\
J_{(2,1)} &= -\sqrt{2}\left(J^{(1,1)}_{(\frac{1}{2},\frac{1}{2},\frac{1}{2})} + J^{(1,1)}_{(-\frac{1}{2},\frac{1}{2},\frac{1}{2})}\right), \qquad & J_{(2,0)} &= -2J^{(1,1)}_{(0,0,0)}\,, \\
J_{(2,-1)} &= -\sqrt{2}\left(J^{(1,1)}_{(-\frac{1}{2},\frac{1}{2},-\frac{1}{2})} - J^{(1,1)}_{(\frac{1}{2},\frac{1}{2},-\frac{1}{2})}\right), \qquad & J_{(2,-2)} &= 2J^{(1,1)}_{(0,1,-1)}\,.
\end{aligned} \tag{III.1.7}$$

They satisfy the commutation relations

$$\left[J_{(L',M')}, J_{(L'',M'')}\right] = -2\sqrt{3}\begin{bmatrix} (1,1) & (1,1) & (1,1)_a \\ L',M' & L'',M'' & L,M \end{bmatrix} J_{(L,M)}. \tag{III.1.8}$$

The coefficient on the right hand side of (III.1.8) are the SU(3) noncanonical Clebsch-Gordan coefficient. The relations between the noncanonical basis state vectors and the canonical state vectors for the adjoint representation $(1,1)$ are similar to the relations of the generators (III.1.7), with the only difference in the normalization factor of 1/2 to keep the state vectors normalized.

The noncanonical Clebsch-Gordan coefficients of SU(3) which are used for the calculation of the SU(3) Skyrme model with the SO(3) soliton are factorized according to

$$\begin{bmatrix} (1,1) & (1,1) & (1,1)_\gamma \\ L',M' & L'',M'' & L,M \end{bmatrix} = \begin{bmatrix} (1,1) & (1,1) & (1,1)_\gamma \\ L' & L'' & L \end{bmatrix}\begin{bmatrix} L' & L'' & L \\ M' & M'' & M \end{bmatrix}, \tag{III.1.9}$$

where the second factor on the right hand side of equation (III.1.9) refers to the well-known CG coefficient of SO(3), and the first factor (the noncanonical isoscalar factor) is independent of M', M'' and M, and differs from the SU(3) isoscalar factors defined in (II.1.8). The label γ in the CG coefficient takes two values $\gamma = 1$ and 2 for antisymmetrical and symmetrical cases. Some isoscalar factors for the symmetrical and antisymmetrical representation are given in Appendix C respectively.

2. Soliton quantization on the SU(3) manifold

We take the SO(3) skyrmion (III.1.3) with $j = 1$ as the classical ground state U_0 for the ansatz. The quantization of the Skyrme model can be achieved by means of the collective coordinates $q^\alpha(t)$

$$U\big(\hat{x}, F(r), q(t)\big) = A\big(q(t)\big)U_0\big(\hat{x}, F(r)\big)A^\dagger\big(q(t)\big), \qquad A\big(q(t)\big) \in \text{SU(3)}. \tag{III.2.10}$$

Like before we shall consider the Skyrme Lagrangian quantum mechanically *ab initio* and eight unconstraint angles $q^\alpha(t)$ to be the quantum variables. Because the ansatz U_0 does not commute with all SU(3) generators the quantization is realized on the eight parameter group manifold. That is to the contrary of the usual seven-dimensional homogeneous space SU(3)/U(1) in the canonical embedded soliton case [75]. The generalized coordinates $q^\beta(t)$ and the velocities $\dot{q}^\alpha\,(t)$ satisfy the commutation relations

$$\left[\dot{q}^\alpha, q^\beta\right] = -\mathrm{i}g^{\alpha\beta}(q), \tag{III.2.11}$$

where the explicit form of the symmetric tensor $g^{\alpha\beta}(q)$ is determined after the quantization condition has been imposed.

For the purpose of defining the metric tensor in the Lagrangian we use an approximate expression

$$A^\dagger \dot{A} \approx \frac{1}{2}\Big\{\dot{q}^\alpha, C'^{(L,M)}_\alpha(q)\Big\} \left\langle \ \middle| J_{(L,M)} \middle| \ \right\rangle + ... \tag{III.2.12}$$

which will be specified later.

After substitution of the ansatz (III.1.3) into the model Lagrangian density (I.3.21), it takes a form (I.7.7) which is quadratic in the velocities $\dot{q}^\alpha$. The metric tensor is

$$g_{\alpha\beta}(q,F) = C'^{(L,M)}_\alpha(q) E_{(L,M)(L',M')}(F) C'^{(L',M')}_\beta(q), \tag{III.2.13}$$

with

$$E_{(L,M)(L',M')}(F) = -(-1)^M a_L(F)\delta_{L,L'}\delta_{M,-M'}. \tag{III.2.14}$$

The soliton moments of inertia are given as integrals over the dimensionless variable $\tilde{r} = ef_\pi r$

$$a_1(F) = \frac{1}{e^3 f_\pi}\frac{8\pi}{3}\int \mathrm{d}\tilde{r}\tilde{r}^2 \sin^2 F \left(1 + F'^2 + \frac{1}{\tilde{r}^2}\sin^2 F\right), \tag{III.2.15a}$$

$$a_2(F) = \frac{1}{e^3 f_\pi}\frac{8\pi}{5}\int \mathrm{d}\tilde{r}\tilde{r}^2\Bigg(\sin^2 F\Big(3 + 2\cos 2F + (9 + 8\cos 2F)F'^2\Big) + (9 + 4\cos 2F)\frac{\sin^2 F}{\tilde{r}^2}\Bigg). \tag{III.2.15b}$$

The first moment of inertia $a_1(F)$ coincides with the SU(2) soliton moment of inertia, however the second moment of inertia $a_2(F)$ differs from the second soliton momenta of inertia (II.3.8c) in the usual SU(3) $\supset$ SU(2) ansatz case. The employment of the noncanonical SU(3) basis leads to the moments (III.2.15) which do not agree with the SO(3) soliton moments of inertia defined in [106].

The canonical momenta are defined as

$$p_\beta = \frac{\partial L}{\partial \dot{q}^\beta} = \frac{1}{2}\left\{\dot{q}^\alpha, g_{\alpha\beta}\right\}. \tag{III.2.16}$$

They are conjugate to the coordinates and satisfy the commutation relations $[p_\beta, q^\alpha] = -\mathrm{i}\delta_{\alpha\beta}$. These relations and the equation (III.2.16) fix the explicit expressions of the functions introduced in (III.2.11)

$$g^{\alpha\beta}(q) = (g_{\alpha\beta})^{-1} = C'^\alpha_{(L,M)}(q) E^{(L,M)(L',M')}(F) C'^\beta_{(L',M')}(q), \tag{III.2.17}$$

where

$$E^{(L,M)(L',M')}(F) = -(-1)^M \frac{1}{a_{(L)}(F)}\delta_{L,L'}\delta_{M,-M'}\,. \tag{III.2.18}$$

Having determined the function $g^{\alpha\beta}(q)$ we can obtain an explicit expression of $A^\dagger\dot{A}$:

$$\begin{aligned} A^\dagger \dot{A} &= A^\dagger\left\{\dot{q}^\alpha, A\right\} \\ &= \frac{1}{2}\Big\{\dot{q}^\alpha, C'^{(LM)}_\alpha(q)\Big\}\left\langle \ \middle| J_{(LM)} \middle| \ \right\rangle \\ &\quad - \frac{\mathrm{i}}{2}E^{(L_1M_1)(L_2M_2)}\left\langle \ \middle| J_{(L_1M_1)}J_{(L_2M_2)} \middle| \ \right\rangle. \end{aligned} \tag{III.2.19}$$

Eight right transformation generators are defined as

$$\hat{R}_{(L,M)} = \frac{\mathrm{i}}{2}\Big\{p_\alpha, C'^{\alpha}_{(L,M)}(q)\Big\} = \frac{\mathrm{i}}{2}\Big\{\dot{q}^\beta, C'^{(L',M')}_{\beta}(q)\Big\}E_{(L',M')(L,M)}. \tag{III.2.20}$$

They satisfy the commutation relations (III.1.8). Here the functions $C'^{\alpha}_{(L,M)}(q)$ are dual to the functions defined in (III.1.1)

$$\sum_\alpha C'^{\alpha}_{(L,M)}(q)C'^{(L',M')}_{\alpha}(q) = \delta_{(L,M)(L',M')}\,, \tag{III.2.21a}$$

$$\sum_{L,M} C'^{\alpha}_{(L,M)}(q)C'^{(L,M)}_{\alpha'}(q) = \delta_{\alpha,\alpha'}\,. \tag{III.2.21b}$$

The action of the right transformation generators on the Wigner matrix of the SU(3) irrep is well defined:

$$\Big[\hat{R}_{(L,M)}, D^{(\lambda,\mu)}_{(\alpha,L',M')(\beta,L'',M'')}(q)\Big] = \left\langle \begin{matrix}(\lambda,\mu)\\ \alpha, L', M'\end{matrix}\middle|\hat{J}_{(L,M)}\middle|\begin{matrix}(\lambda,\mu)\\ \alpha_0, L_0, M_0\end{matrix}\right\rangle \times D^{(\lambda,\mu)}_{(\alpha_0,L_0,M_0)(\beta,L'',M'')}(q). \tag{III.2.22}$$

The indices α and β label the multiplets of (L,M). The substitution of the ansatz (III.2.10) into the model Lagrangian density (I.3.21) and integration over the spatial coordinates leads to the effective Lagrangian of the form

$$L_{\text{eff}} = \frac{1}{2a_2(F)}(-1)^M\hat{R}_{(L,M)}\hat{R}_{(L,-M)} + \left(\frac{1}{2a_1(F)} - \frac{1}{2a_2(F)}\right) \times (-1)^m\left(\hat{R}_{(1,m)}\cdot\hat{R}_{(1,-m)}\right) - M_{\text{cl}} - \Delta M_1 - \Delta M_2 - \Delta M_3, \tag{III.2.23}$$

where M_{cl} is the classical skyrmion mass, and $\Delta M_k = \int \mathrm{d}^3x\Delta\mathcal{M}_k(F)$ are the quantum corrections to the Lagrangian:

$$\Delta\mathcal{M}_1 = -\frac{2\sin^2 F}{a_1^2(F)}\left(f_\pi^2\,(2-\cos 2F) + \frac{3}{e^2}\left(F'^2\,(2+\cos 2F) + \frac{\sin^2 F}{r^2}\right)\right); \tag{III.2.24a}$$

$$\Delta\mathcal{M}_2 = -\frac{2\sin^2 F}{a_2^2(F)}\left(f_\pi^2\,(14+11\cos 2F) + \frac{3}{e^2}\Big(F'^2\,(42+41\cos 2F) + (25+12\cos 2F)\,\frac{\sin^2 F}{r^2}\Big)\right); \tag{III.2.24b}$$

$$\Delta\mathcal{M}_3 = -\frac{4\sin^2 F}{a_1(F)\,a_2(F)}\left(f_\pi^2\,(4+\cos 2F) + \frac{3}{e^2}\Big(F'^2\,(6+5\cos 2F) + (1-\cos 2F)\,\frac{\sin^2 F}{r^2}\Big)\right). \tag{III.2.24c}$$

Two operators in (III.2.23) are the quadratic Casimir operators of the SU(3) and SO(3) groups:

$$\hat{C}_2^{\text{SU(3)}} = (-1)^A J^{(1,1)}_{(A)}J^{(1,1)}_{(-A)} = \frac{1}{4}(-1)^m R_{(1,m)}R_{(1,-m)} + \frac{1}{4}(-1)^M R_{(2,M)}R_{(2,-M)},$$
$$\hat{C}_2^{\text{SO(3)}} = (-1)^m R_{(1,m)}R_{(1,-m)} \neq \hat{C}_2^{\text{SU(2)}}. \tag{III.2.25}$$

A simple structure of the operators permits to write down the eigenfunctions in the next section.

3. The structure of the Hamiltonian density and the symmetry breaking term

To find the explicit expression of the Lagrangian and the Hamiltonian density of the quantum skyrmion we take into account the explicit commutation relations (III.2.11) and (III.2.17). A lengthy manipulation with the ansatz yields the expression of the Lagrangian density

$$\mathcal{L}_{\text{Sk}} = \mathcal{K} - \mathcal{M}_{\text{cl}} - \Delta\mathcal{M}_1 - \Delta\mathcal{M}_2 - \Delta\mathcal{M}_3, \tag{III.3.26}$$

where the kinetic (operator) part of the Lagrangian density is as follows

$$\begin{aligned}\mathcal{K} = {}& \frac{4}{a_L^2(F)}(-1)^M \hat{R}_{(L,M)}\hat{R}_{(L,M')}\Bigg\{\frac{f_\pi^2}{4}\left(\delta_{-M,M'} - D^L_{-M,M'}(\hat{x},F(r))\right) \\ & + \frac{3}{e^2}\left(\delta_{-M,M'} - D^L_{-M,M'}(\hat{x},F(r))\right)\Bigg\{\left(F'^2 - \frac{1}{r^2}\sin^2 F\right) \\ & \times \frac{2\sqrt{\pi}(2L+1)\sqrt{\frac{1}{2}l+1}}{\sqrt{3}(5-2L)\sqrt{2l+1}}(-1)^{L+M+\frac{1}{2}l+1}\begin{Bmatrix} 1 & 1 & l \\ L & L & L\end{Bmatrix} \\ & \times \begin{bmatrix} L & L & l \\ M & M' & m\end{bmatrix} Y_{l,m}(\theta,\varphi) + \frac{1}{r^2}\sin^2 F\frac{1}{(5-2L)}\,\delta_{-M,M'}\Bigg\}\Bigg\}.\end{aligned} \tag{III.3.27}$$

Here the Wigner $D^L_{M,M'}(\hat{x},F(r))$ matrices are used which in fact are the hedgehog anzatz for the irrep $L = 1$ and 2 in (III.1.3). The maximal l of the spherical functions $Y_{l,m}(\theta,\varphi)$ which appear in (III.3.27) are $l = 4$.

We define the momentum density as $\mathcal{P}_\beta = \frac{\partial\mathcal{L}}{\partial\dot{q}^\beta}$. The kinetic energy density is defined as $2\mathcal{K} = \frac{1}{2}\left\{\mathcal{P}_\beta,\dot{q}^\beta\right\}$. Thus the Skyrme model Hamiltonian density takes the form

$$\mathcal{H} = \frac{1}{2}\left\{\mathcal{P}_\beta,\dot{q}^\beta\right\} - \mathcal{L}_{\text{Sk}} = \mathcal{K} + \mathcal{M}_{\text{cl}} + \Delta\mathcal{M}_1 + \Delta\mathcal{M}_2 + \Delta\mathcal{M}_3. \tag{III.3.28}$$

The operator (kinetic) part of the Lagrangian (III.2.23) and the kinetic part of the corresponding Hamiltonian depend on the quadratic Casimir operators of the SU(3) and SO(3) groups which are constructed using the right transformation generators (III.2.20). The eigenstates of the Hamiltonian $H = \int \mathrm{d}^3x\mathcal{H}$ are

$$\left|\begin{matrix}(\Lambda,\Theta)\\ \alpha,S,N;\beta,S',N'\end{matrix}\right\rangle = \sqrt{\dim(\Lambda,\Theta)}D^{*(\Lambda,\Theta)}_{(\alpha,S,N)(\beta,S',N')}(q)\left|0\right\rangle, \tag{III.3.29}$$

where the complex conjugate Wigner matrix elements of the (Λ,Θ) representation depend on eight quantum variables q^β. The indices α and β label the multiplets of the SO(3) group. $|0\rangle$ denotes the vacuum state. Due to the structure of the density operator (III.3.27), the noncanonical soliton mass distribution has a complex but well defined tensorial structure which depends on radial functions $F(r)$ and the spherical harmonics $Y_{l,m}(\theta,\varphi)$ of order $l = 1,2,3,4$.

The mass or energy functional of the equation state is following

$$\begin{aligned}M = {}& \frac{2}{3a_2(F)}\left(\Lambda^2 + \Theta^2 + \Lambda\Theta + 3\Lambda + 3\Theta\right) \\ & + \left(\frac{1}{2a_1(F)} - \frac{1}{2a_2(F)}\right)S(S+1) + M_{\text{cl}} + \Delta M_1 + \Delta M_2 + \Delta M_3\,.\end{aligned} \tag{III.3.30}$$

In contrast to the positive impact of the Casimir operators (quantum rotation) to the classical mass $M_{\rm cl}$, the quantum corrections ΔM which appear from the commutation relations are negative.

We take into account the chiral symmetry breaking effects by introducing a term

$$\mathcal{M}_{\rm SB} = \frac{1}{4N} f_\pi m_0^2 \,\mathrm{Tr}\Big(U + U^\dagger - 2\Big), \tag{III.3.31}$$

which takes an explicit form

$$\mathcal{M}_{\rm SB} = \frac{1}{2} f_\pi m_0^2 \sin^2 F. \tag{III.3.32}$$

In (III.3.31) we used the same normalization factor $N = 4$ which is defined for the SO(3) classical soliton $j = 1$.

A direct calculation shows that the Wess-Zumino term $L_{\rm WZ}$ is equal to zero for the noncanonically embedded SO(3) soliton.

4. Summary

In this chapter we have considered a new ansatz for the Skyrme model which is the noncanonically embedded SU(3) $\supset$ SO(3) soliton. The strict canonical quantization of the soliton leads to new expressions of the moments of inertia and the negative quantum corrections ΔM. The quantum corrections which appear from the commutation relations compensate the effect of the positive SU(3) and SO(3) "rotation" kinetic energy. The variation of the quantum energy functional (III.3.30) allow to find the stable solutions of quantum skyrmions even without the symmetry breaking term. The shape of the quantum skyrmion is not fixed like in the semiclassical "rigid body" case and the infinite tower of solutions for the higher representations (Λ, Θ) is absent. It means that the "fast rotation" destroys the quantum skyrmion. The unitary field $U(\mathbf{x}, t)$ for the SU(3) Skyrme model can be defined in an arbitrary irrep (λ, μ). The SU(2) and SO(3) ansatzes can be constructed as the reducible representations of the SU(2) embedded into the SU(3) irrep (λ, μ) in different ways. It can generate different types of quantum skyrmions.

IV. Noncanonically embedded rational map soliton in the quantum SU(3) Skyrme model

The aim of this chapter is to discuss the group-theoretical aspects of the canonical quantization of the SU(3) Skyrme model in the rational map antsatz approximation with the baryon number $B \geq 2$. The ansatz is defined in the noncanonical SU(3) $\supset$ SO(3) bases as an SO(3) solitonic solution. The canonical quantization generates five different moments of inertia. The proposed ansatz can be used to describe light nuclei as special skyrmions.

1. Noncanonical embedding of the rational map soliton

The Skyrme model is a Lagrangian density for a unitary field $U(\mathbf{x},t)$ that belongs to the general representation of the SU(3) group [14]. We consider the unitary field in the fundamental representation (1,0) of the SU(3) group. The chirally symmetric Lagrangian density has the standard form (I.3.21), where the "right" and "left" chiral currents are defined as in (III.1.1) and have the values on the SU(3) algebra. Explicit expressions of functions $C_i^{(L,M)}(\alpha)$ and $C_i^{\prime(L,M)}(\alpha)$ depend on the group parametrization α^i. The noncanonical SU(3) generators may be expressed in terms of the canonical generators $J^{(1,1)}_{(Z,I,M)}$ defined in (III.1.7). The canonical generators satisfy (III.1.8) commutation relations. The state vectors for the canonical bases SU(3) $\supset$ SU(2) and the non canonical bases SU(3) $\supset$ SO(3) are equivalent in fundamental representation.

We suggest to use the rational map antsatz in the SO(3) case as a matrix

$$\begin{aligned}(U_R)_{M,M'} &= D^1_{M,M'}(\varkappa) = \big(\exp\big(2\mathrm{i}\hat{n}_a J_{(1,a)} F(r)\big)\big)_{M,M'} \\ &= 2\sin^2 F(-1)^M \hat{n}_{-M}\hat{n}_{M'} + \mathrm{i}\sqrt{2}\sin 2F \begin{bmatrix} 1 & 1 & 1 \\ M & u & M' \end{bmatrix} \hat{n}_u \\ &\quad + \cos 2F\, \delta_{M,M'}, \end{aligned} \tag{IV.1.1}$$

where the unit vector $\hat{\mathbf{n}}$ is defined [49] in terms of a rational complex function $R(z)$ (I.4.3). The triplet $\varkappa$ of the Euler angles is defined by $\hat{n}$ and $F(r)$. By differentiation of $\hat{\mathbf{n}}$ we get an expression which gives the advantage of the following calculations

$$(-1)^s(\nabla_{-s} r\hat{n}_m)(\nabla_s r\hat{n}_{m'}) = \hat{n}_m\hat{n}_{m'} + \mathcal{I}\big((-1)^m\delta_{-m,m'} - \hat{n}_m\hat{n}_{m'}\big) \tag{IV.1.2}$$

where the symbol $\mathcal{I}$ is previously mentioned function (I.4.13) that solely depends on the angles θ and φ.

The baryon charge density for the rational map skyrmion is expressed as

$$\mathcal{B}(r,\theta,\varphi) = \frac{1}{24N\pi^2}\epsilon^{0k\ell m}\,\mathrm{Tr}\Big(R_k R_\ell R_m\Big) = -\frac{2\mathcal{I}\,(\theta,\varphi)}{N\pi^2}\frac{F'(r)\sin^2 F}{r^2}\,. \tag{IV.1.3}$$

Since this expression contains the $\mathcal{I}$ function, there is no need to modify the usual boundary conditions $F(0) = \pi$ and $F(\infty) = 0$ to account for the profile function. The integral by the spatial angles of $\mathcal{I}$ is proportional to the baryon number [88]

$$\int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\theta\, \mathcal{I} \sin\theta = 4\pi B\,. \tag{IV.1.4}$$

With the ansatz (IV.1.1) the Lagrangian density (I.3.21) reduces to the classical Skyrme Lagrangian for any baryon number B

$$\begin{aligned}\mathcal{L}_{\mathrm{cl}}(r,\theta,\varphi) = -\,\mathcal{M}_{\mathrm{cl}} = -N\Bigg(f_\pi^2 \left(\frac{F'^2(r)}{2} + \frac{\mathcal{I}\sin^2 F}{r^2} \right) \\ - \frac{1}{e^2}\frac{\mathcal{I}\sin^2 F}{r^2}\left(F'^2(r) + \frac{\mathcal{I}\sin^2 F}{2r^2} \right)\Bigg),\end{aligned} \tag{IV.1.5}$$

which describes the skyrmion mass density. The normalization factor $N = 4$ is chosen requiring that in the fundamental representation of SO(3) group for the spherically symmetric skyrmion case the baryon number equals unity.

After introduction of the dimensionless coordinate $\tilde{r} = e f_\pi r$, the variation of the Lagrangian yields the following differential equation for the profile function

$$\begin{aligned}F''(\tilde{r})\Big(1 + \frac{2B\sin^2 F(\tilde{r})}{\tilde{r}^2}\Big) + \frac{2F'(\tilde{r})}{\tilde{r}} + \frac{F'^2(\tilde{r})B\sin 2F(\tilde{r})}{\tilde{r}^2} \\ - \frac{B\sin 2F(\tilde{r})}{\tilde{r}^2} - \frac{I_2\sin^2 F(\tilde{r})\sin 2F(\tilde{r})}{\tilde{r}^4} = 0\,.\end{aligned} \tag{IV.1.6}$$

Here an abbreviation is used:

$$I_2 = \frac{1}{4\pi}\int_0^{2\pi}\mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\theta \mathcal{I}^2 \sin\theta\,. \tag{IV.1.7}$$

In the limit of $\tilde{r} \to \infty$, the equation (IV.1.6) reduces to a simple asymptotic form

$$F''(\tilde{r}) + \frac{2F'(\tilde{r})}{\tilde{r}} - \frac{2BF(\tilde{r})}{\tilde{r}^2} = 0\,. \tag{IV.1.8}$$

From this the asymptotic large distance solution, which satisfies physical boundary conditions, can be easily obtained as

$$F(\tilde{r}) = C_1 \tilde{r}^{-\frac{1+\sqrt{1+8B}}{2}}\,. \tag{IV.1.9}$$

Here C_1 is a constant that is determined by a continuous joining of the numerical small distance solution onto the analytic asymptotic solution. Equations (IV.1.5–IV.1.9) are valid for all B, provided that the corresponding function $\mathcal{I}$ is used.

2. Canonical quantization of the Soliton

In this work the quantization of the model is performed by the analogous quantization methodic described in chapter III. The collective coordinates are employed for the separation of the variables in the unitary field U_R (IV.1.1), which depend on the temporal and spatial coordinates

$$U\big(\hat{n}, F(r), q(t)\big) = A\big(q(t)\big) U_R\big(\hat{n}, F(r)\big) A^\dagger\big(q(t)\big), \qquad A\big(q(t)\big) \in \mathrm{SU(3)}. \tag{IV.2.1}$$

The same as previously, the Skyrme Lagrangian is considered quantum mechanically *ab initio* in contrast to the conventional semiclassical quantization of the soliton as a rigid body. The generalized coordinates $q^i(t)$ and the corresponding velocities $\dot{q}^i(t)$ satisfy commutation relations (III.2.11). Due to the Weyl operator ordering (I.7.4) no further ordering ambiguity appears in the Lagrangian or the Hamiltonian. The differentiation of the arbitrary q-dependent unitary matrix $G(q)$ is expressed in terms of functions $C'^{(L,M)}_{\alpha}$ and the matrix elements of the group generator $J_{(L,M)}$

$$\frac{\partial}{\partial q^\alpha} G^{(\lambda,\mu)}_{(A)(B)}(q) = C'^{(L,M)}_{\alpha}(q) G^{(\lambda,\mu)}_{(A)(A')}(q) \left\langle \begin{matrix}(\lambda,\mu)\\ A'\end{matrix} \middle| J_{(L,M)} \middle| \begin{matrix}(\lambda,\mu)\\ (B)\end{matrix} \right\rangle. \tag{IV.2.2}$$

After substitution of the ansatz (IV.2.1) into the model Lagrangian density (I.3.21) and integration over spatial coordinates, the Lagrangian has this form:

$$L = \frac{1}{2}\dot{q}^\alpha g_{\alpha\beta}(q,F)\dot{q}^\beta + a^0 \frac{1}{2}\left\{\dot{q}^\alpha, C'^{(2,0)}_{\alpha}(q)\right\} + \left[(\dot{q})^0 - \text{order terms}\right], \tag{IV.2.3}$$

where the metric tensor $g_{\alpha\beta}(q,F)$ and the intermediate function $E_{(L,M)(L',M')}$ is defined as in eq. (III.2.13) and eq. (III.2.14). Note that the exact expression of the coefficient a^0 is not important for the calculation of $g_{\alpha\beta}$. There are five different quantum moments of inertia in (IV.2.3):

$$a_{(1,0)}(F) = \frac{1}{e^3 f_\pi}\int \mathrm{d}^3\tilde{r}\tilde{r}^2 \sin^2 F \left(n_0^2 - 1\right)\left(1 + F'^2 + \frac{\mathcal{I}}{r^2}\sin^2 F\right); \tag{IV.2.4a}$$

$$a_{(1,1)}(F) = a_{(1,-1)}(F) = \frac{1}{2e^3 f_\pi}\int \mathrm{d}^3\tilde{r}\tilde{r}^2 \sin^2 F \left(n_0^2 + 1\right)\left(1 + F'^2 + \frac{\mathcal{I}}{r^2}\sin^2 F\right); \tag{IV.2.4b}$$

$$\begin{aligned} a_{(2,0)}(F) = &\frac{1}{e^3 f_\pi}\int \mathrm{d}^3\tilde{r}\tilde{r}^2 \sin^2 F \left(n_0^2 - 1\right)\Big(\cos^2 F + n_0^2 \sin^2 F \\ &- \left(n_0^2 - 4\cos^2 F + 2n_0^2 \cos 2F\right) F'^2 \\ &+ \frac{\mathcal{I}}{r^2}\sin^2 F \left(2\cos^2 F + n_0^2(4 - \cos 2F)\right)\Big); \end{aligned} \tag{IV.2.4c}$$

$$\begin{aligned} a_{(2,1)}(F) = a_{(2,-1)}(F) = &\frac{1}{2e^3 f_\pi}\int \mathrm{d}^3\tilde{r}\tilde{r}^2 \sin^2 F \Big(3 + 2\cos 2F - 3n_0^2 + 4n_0^4 \sin^2 F \\ &+ \left(9 + 8\cos 2F - 3n_0^2 - 4n_0^4(1 + 2\cos 2F)\right) F'^2 \\ &+ \frac{\mathcal{I}}{r^2}\sin^2 F \left(9 + 4\cos 2F - 15n_0^2 + 4n_0^4(4 - \cos 2F)\right)\Big); \end{aligned} \tag{IV.2.4d}$$

$$\begin{aligned} a_{(2,2)}(F) = a_{(2,-2)}(F) = &\frac{1}{4e^3 f_\pi}\int \mathrm{d}^3\tilde{r}\tilde{r}^2 \sin^2 F \Big(-3 - \cos 2F - 12n_0^2 \cos^2 F + 2n_0^4 \sin^2 F \\ &- 2\left(3 + 2\cos 2F - 24n_0^2 \cos^2 F + n_0^4(1 + 2\cos 2F)\right) F'^2 \\ &- \frac{2\mathcal{I}}{r^2}\sin^2 F \left(6 + \cos 2F - 12n_0^2 \cos^2 F - n_0^4(4 - \cos 2F)\right)\Big); \end{aligned} \tag{IV.2.4e}$$

where $\mathrm{d}^3\tilde{r} = \sin\theta \mathrm{d}\theta \mathrm{d}\varphi \mathrm{d}\tilde{r}$. These quantum moments depend on the profile function $F(r)$, one component of the rational map vector n_0 and the function $\mathcal{I}(\theta,\varphi)$.

The canonical momenta are defined as

$$p_\beta = \frac{\partial L}{\partial \dot{q}^\beta} = \frac{1}{2}\left\{\dot{q}^\alpha, g_{\alpha\beta}\right\} + a^0 C'^{(2,0)}_{\beta}(q). \tag{IV.2.5}$$

Note that the momenta do not commute and have terms which do not contain velocity. The parametrization q^α of the group manifold is significant for the definition of the canonical momenta. For the time being we do not require $[p_\alpha, p_\beta] = 0$. The momenta and the conjugate coordinates satisfy the commutation relations $[p_\beta, q^\alpha] = -\mathrm{i}\delta_{\alpha\beta}$. These commutation relations determine the explicit expressions of the functions $g^{\alpha\beta}(q) = (g_{\alpha\beta})^{-1}$.

Determination of functions $g^{\alpha\beta}(q)$ allows us to obtain an explicit expression of (III.2.12):

$$\begin{aligned}
A^\dagger \dot{A} &= A^\dagger \left\{\dot{q}^\alpha, A\right\} \\
&= \frac{1}{2}\Big\{\dot{q}^\alpha, C'^{(L,M)}_\alpha(q)\Big\} \left\langle \, \middle| J_{(L,M)} \middle| \, \right\rangle - \frac{\mathrm{i}}{2} E^{(L_1,M_1)(L_2,M_2)} \left\langle \, \middle| J_{(L_1,M_1)} J_{(L_2,M_2)} \middle| \, \right\rangle \\
&= \frac{1}{2}\Big\{\dot{q}^\alpha, C'^{(L,M)}_\alpha(q)\Big\} \left\langle \, \middle| J_{(L,M)} \middle| \, \right\rangle + \frac{\mathrm{i}}{2a_0} \cdot \mathbb{1} - \frac{\mathrm{i}}{2a_2} \left\langle \, \middle| J_{(2,0)} \middle| \, \right\rangle ,
\end{aligned} \tag{IV.2.6}$$

where a_0 and a_2 are constructed from the quantum moments of inertia:

$$\frac{1}{a_0} = \frac{1}{3}\left(\frac{2}{a_{(1,0)}} + \frac{4}{a_{(1,1)}} + \frac{2}{a_{(2,0)}} + \frac{4}{a_{(2,1)}} + \frac{4}{a_{(2,2)}}\right); \tag{IV.2.7a}$$

$$\frac{1}{a_2} = \frac{1}{\sqrt{3}}\left(-\frac{1}{a_{(1,0)}} + \frac{1}{a_{(1,1)}} + \frac{1}{a_{(2,0)}} + \frac{1}{a_{(2,1)}} - \frac{2}{a_{(2,2)}}\right). \tag{IV.2.7b}$$

The field (IV.2.1) is substituted in the Lagrangian density (I.3.21) in order to obtain the explicit expression in terms of the collective coordinates and the space coordinates. After long calculation by using (IV.2.6) and the commutation relation (III.2.11), we get a complete explicit expression of the Skyrme model Lagrangian density

$$\begin{aligned}
\mathcal{L}(q, \dot{q}, \varkappa) = &\Big\{\dot{q}^\alpha, C'^{(L,M_1)}_\alpha(q)\Big\}\Big\{\dot{q}^\beta, C'^{(L,M_2)}_\beta(q)\Big\}\mathcal{V}_1(\varkappa) \\
&+ \mathrm{i}\Big\{\dot{q}^\alpha, C'^{(L,M_1)}_\alpha(q)\Big\}\mathcal{V}_2(\varkappa) + \mathcal{V}_3(\varkappa) - \mathcal{M}_{\mathrm{cl}}.
\end{aligned} \tag{IV.2.8}$$

The function $\mathcal{V}_1$ in first term results from the trace of two group generators (see (C.4) below)

$$\begin{aligned}
\mathcal{V}_1(\varkappa) = &\frac{f_\pi^2}{4}(-1)^{M_1}\left(D^L_{-M_1,M_2}(\varkappa) + \overset{+}{D}{}^L_{-M,M'}(\varkappa) - 2\delta_{-M_1,M_2}\right) \\
&+ \frac{1}{16e^2}(-1)^{M'} B_{m,m'}(\varkappa)\frac{3}{5-2L}\begin{bmatrix} L & 1 & L \\ M_1' & m & M' \end{bmatrix}\begin{bmatrix} L & 1 & L \\ M_2' & m' & -M' \end{bmatrix} \\
&\times\left(2\delta_{M_1,M_1'} D^L_{M_2,M_2'}(\varkappa) - \delta_{M_1,M_1'}\delta_{M_2,M_2'} - D^L_{M_1,M_1'}(\varkappa) D^L_{M_2,M_2'}(\varkappa)\right).
\end{aligned} \tag{IV.2.9}$$

The function $\mathcal{V}_2$ results from the trace containing three group generators (see (C.5))

$$\begin{aligned}
\mathcal{V}_2(\varkappa) = &\frac{f_\pi^2}{4}\Bigg(\frac{(-1)^{M_2+M_1}}{a_{(L,M_2)}}\frac{\sqrt{2}}{\sqrt{3}}\sqrt{L^2+L+1}\begin{bmatrix} L & L & 2 \\ M_2 & -M_2' & M_1 \end{bmatrix} \\
&\times\left(D^L_{M_2',M_2}(\varkappa) - \overset{+}{D}{}^L_{M_2',M_2}(\varkappa)\right) - \frac{1}{a_2}\left(D^2_{0,M_1}(\varkappa) - \overset{+}{D}{}^2_{0,M_1}(\varkappa)\right)\Bigg) \\
&+ \frac{1}{4e^2}(-1)^{M_1'+M_2}\frac{\sqrt{2\cdot 3}}{a_{(L,M_2)}}\frac{\sqrt{L^2+L+1}}{\sqrt{5-2L}} B_{m,m'}(\varkappa)\begin{bmatrix} 2 & 1 & 2 \\ M_1'' & m & M_1' \end{bmatrix} \times
\end{aligned}$$

$$\times \begin{bmatrix} L & 1 & L \\ M_2'' & m' & M_2' \end{bmatrix} \begin{bmatrix} L & L & 2 \\ M_2' & M_2''' & -M_1' \end{bmatrix} \Big(\delta_{M_1,M_1''} \delta_{M_2,M_2''} \overset{+}{D}{}^{L}_{M_2''',-M_2'}(\varkappa)$$
$$- \delta_{M_1,M_1''} \delta_{M_2,M_2''} \delta_{-M_2,M_2'''} - \delta_{M_1,M_1''} \delta_{-M_2,M_2'''} D^{L}_{M_2'',M_2}(\varkappa)$$
$$+ \delta_{M_2,M_2''} \delta_{-M_2,M_2'''} \overset{+}{D}{}^{2}_{M_1'',M_1}(\varkappa) \Big). \tag{IV.2.10}$$

And the function $\mathcal{V}_3$ results from the trace containing four group generators (C.6)

$$\mathcal{V}_3(\varkappa) = \frac{f_\pi^2}{4} \frac{4(2L_1+1)(2L_2+1)}{a_{(L_1,M_1)} a_{(L_2,M_2)}} \begin{Bmatrix} L_1 & L_2 & k \\ 1 & 1 & 1 \end{Bmatrix}^2 \begin{bmatrix} L_1 & L_2 & k \\ M_1 & M_2 & u \end{bmatrix}^2 D^{k}_{u,u}(\varkappa) + \frac{3}{a_0^2}$$
$$+ \frac{1}{a_2^2} \Big(1 + \overset{+}{D}{}^{2}_{0,0}(\varkappa) \Big) - \frac{4}{a_{(L,M)}} \Bigg(\frac{1}{a_0} D^{L}_{M,M}(\varkappa) - (-1)^M \frac{1}{a_2} \frac{\sqrt{L^2+L+1}}{\sqrt{2\cdot 3}}$$
$$\times \begin{bmatrix} L & L & 2 \\ M & -M & 0 \end{bmatrix} D^{L}_{M,M}(\varkappa) \Bigg)$$
$$- \frac{3}{2e^2} \frac{(2L_1+1)(2L_2+1)}{\sqrt{(5-2L_1)\,(5-2L_2)}} (-1)^{M_1+M_2+u} B_{m,m'}(\varkappa) \begin{Bmatrix} L_1 & L_1 & k \\ 1 & 1 & 1 \end{Bmatrix}$$
$$\times \begin{Bmatrix} L_2 & L_2 & k \\ 1 & 1 & 1 \end{Bmatrix} \begin{bmatrix} L_1 & 1 & L_1 \\ M_1' & m & M_1'' \end{bmatrix} \begin{bmatrix} L_2 & 1 & L_2 \\ M_2' & m' & M_2'' \end{bmatrix} \begin{bmatrix} L_1 & L_1 & k \\ M_1 & -M_1'' & u \end{bmatrix}$$
$$\times \begin{bmatrix} L_2 & L_2 & k \\ -M_2 & M_2'' & u \end{bmatrix} \Bigg(\frac{1}{a_{(L_1,M_1)} a_{(L_2,M_2)}} \Big(\delta_{M_2,M_2'} D^{L_1}_{M_1',M_1}(\varkappa) \left(1-(-1)^k\right)$$
$$- \delta_{M_1,M_1'} \delta_{M_2,M_2'} - D^{L_1}_{M_1',M_1}(\varkappa) D^{L_2}_{M_2',M_2}(\varkappa) \Big)$$
$$+ \frac{1}{a_{(L_1,M_1')} a_{(L_2,M_2)}} \delta_{M_2,M_2'} D^{L_1}_{M_1',M_1}(\varkappa) \left(1+(-1)^k\right) \Bigg). \tag{IV.2.11}$$

Here $B_{m,m'}(\varkappa)$ are

$$B_{m,m'}(\varkappa) = 8(-1)^{m+m'} \hat{n}_{-m} \hat{n}_{-m'} \left(\frac{1}{r^2} \mathcal{I} \sin^2 F - F'^2 \right)$$
$$- (-1)^m \delta_{m,-m'} \frac{8}{r^2} \mathcal{I} \sin^2 F. \tag{IV.2.12}$$

The terms with functions $\mathcal{V}_2$ and $\mathcal{V}_3$ are absent in the semiclassical quantization.

Integration of (IV.2.8) over the space variables gives the Lagrangian

$$L = \frac{1}{8} \Big\{ \dot{q}^\alpha, C'^{(L_1,M_1)}_{\alpha}(q) \Big\} E_{(L_1,M_1)(L_2,M_2)} \Big\{ \dot{q}^\beta, C'^{(L_2,M_2)}_{\beta}(q) \Big\}$$
$$+ \mathrm{i} \Big\{ \dot{q}^\alpha, C'^{(2,0)}_{\alpha}(q) \Big\} V_2 + V_3 - M_{\mathrm{cl}}, \tag{IV.2.13}$$

where $V_i = \int \mathrm{d}^3 x \mathcal{V}_i(\varkappa)$.

The canonical momenta p_β and velocities $\dot{q}^\alpha$ satisfy the following relations:

$$\frac{1}{2} \left\{ p_\beta, f^{\alpha\beta} \right\} = \dot{q}^\alpha + 2\mathrm{i} E^{(2,0)(L,M)} C'^{\alpha}_{(L,M)}(q) V_2$$
$$= \dot{q}^\alpha - \mathrm{i} \frac{2}{a_{(2,0)}} C'^{\alpha}_{(2,0)}(q). \tag{IV.2.14}$$

$$\begin{aligned}\frac{1}{2}\Big\{\dot{q}^{\alpha}, C'^{(L,M)}_{\alpha}(q)\Big\} &= E^{(L,M)(L',M')}\frac{1}{2}\left\{p_\beta, C'^{\beta}_{(L',M')}(q)\right\} - 2\mathrm{i}E^{(2,0)(L,M)}V_2\\ &= E^{(L,M)(L',M')}\frac{1}{2}\left\{p_\beta, C'^{\beta}_{(L',M')}(q)\right\} + \mathrm{i}\frac{2}{a_{(2,0)}}V_2. \qquad \text{(IV.2.15)}\end{aligned}$$

It is possible to choose the parametrization on the SU(3) group manifold so that the eight operators

$$\begin{aligned}\hat{R}_{(L,M)} &= \frac{\mathrm{i}}{2}\left\{p_\beta, C'^{\beta}_{(L,M)}(q)\right\}\\ &= \frac{\mathrm{i}}{2}E_{(L,M)(L',M')}\left\{\dot{q}^{\alpha}, C'^{(L',M')}_{\alpha}(q)\right\} - 2\delta_{(2,0)(L,M)}V_2, \qquad \text{(IV.2.16)}\end{aligned}$$

are defined as the group generators satisfying the commutation relations

$$\left[\hat{R}_{(L_1,M_1)}, \hat{R}_{(L_2,M_2)}\right] = -2\sqrt{3}\begin{bmatrix}(1,1) & (1,1) & (1,1)_a\\ (L_1,M_1) & (L_2,M_2) & (L,M)\end{bmatrix}\hat{R}_{(L,M)}. \qquad \text{(IV.2.17)}$$

It is easy to check that because of the choice (IV.2.16) the requirement $[p_\alpha, p_\beta] = 0$ is satisfied for certain. The proof for the SU(2) group can be found in [93]. The generators (IV.2.16) act on the Wigner matrix of the SU(3) irreducible representation as right transformation generators:

$$\begin{aligned}&\left[\hat{R}_{(L,M)}, D^{(\lambda,\mu)}_{(\alpha_1L_1M_1)(\alpha_2L_2M_2)}(q)\right] = D^{(\lambda,\mu)}_{(\alpha_1L_1M_1)(\alpha_2L_2M_2)}(q)\left\langle {(\lambda,\mu) \atop (\alpha_2'L_2'M_2')}\right| J_{(L,M)}\left| {(\lambda,\mu) \atop (\alpha_2L_2M_2)}\right\rangle;\\ &\left[\hat{R}_{(L,M)}, \overset{+}{D}{}^{(\lambda,\mu)}_{(\alpha_1L_1M_1)(\alpha_2L_2M_2)}(q)\right] = -\left\langle {(\lambda,\mu) \atop (\alpha_1L_1M_1)}\right| J_{(L,M)}\left| {(\lambda,\mu) \atop (\alpha_1'L_1'M_1')}\right\rangle \overset{+}{D}{}^{(\lambda,\mu)}_{(\alpha_1'L_1'M_1')(\alpha_2L_2M_2)}(q).\end{aligned} \qquad \text{(IV.2.18)}$$

The indices α_1 and α_2 label the multiplets of (L, M). The right transformation generators satisfy the relation

$$\begin{aligned}E^{(L,M)(L',M')}\hat{R}_{(L,M)}\hat{R}_{(L',M')} &= \frac{1}{a_{(1,0)}}\hat{R}_{(1,0)}\hat{R}_{(1,0)} + \frac{1}{a_{(2,0)}}\hat{R}_{(2,0)}\hat{R}_{(2,0)}\\ &- \frac{1}{a_{(1,1)}}\left(\hat{R}_{(1,1)}\hat{R}_{(1,-1)} + \hat{R}_{(1,-1)}\hat{R}_{(1,1)}\right) - \frac{1}{a_{(2,1)}}\left(\hat{R}_{(2,1)}\hat{R}_{(2,-1)} + \hat{R}_{(2,-1)}\hat{R}_{(2,1)}\right)\\ &+ \frac{1}{a_{(2,2)}}\left(\hat{R}_{(2,2)}\hat{R}_{(2,-2)} + \hat{R}_{(2,-2)}\hat{R}_{(2,2)}\right). \qquad \text{(IV.2.19)}\end{aligned}$$

The left transformation generators are defined as

$$\hat{L}_{(L,M)} = \frac{\mathrm{i}}{2}\left\{p_\beta, C^{\beta}_{(L,M)}(q)\right\}. \qquad \text{(IV.2.20)}$$

From (IV.2.13) we specify the coefficient $a^0 = 2\mathrm{i}V_2$ that was undetermined in (IV.2.5) and derive the Hamiltonian in a form

$$\begin{aligned}H &= \frac{1}{8}\Big\{\dot{q}^{\alpha}, C'^{(L_1,M_1)}_{\alpha}(q)\Big\}E_{(L_1,M_1)(L_2,M_2)}\Big\{\dot{q}^{\beta}, C'^{(L_2,M_2)}_{\beta}(q)\Big\} - V_3 + M_{\mathrm{cl}}\\ &= -\frac{1}{2}\hat{R}_{(L_1,M_1)}E^{(L_1,M_1)(L_2,M_2)}\hat{R}_{(L_2,M_2)} - \frac{2V_2}{a_{(2,0)}}\hat{R}_{(2,0)} - 2\left(\frac{V_2}{a_{(2,0)}}\right)^2 - V_3 + M_{\mathrm{cl}}. \qquad \text{(IV.2.21)}\end{aligned}$$

We define the state vectors as the complex conjugate Wigner matrix elements of the (Λ, Θ) representation depending on eight quantum variables q^α:

$$\left| \begin{matrix} (\Lambda, \Theta) \\ \alpha, S, N; \beta, S', N' \end{matrix} \right\rangle = \sqrt{\dim(\Lambda, \Theta)} D^{*(\Lambda,\Theta)}_{(\alpha,S,N)(\beta,S',N')}(q) \left|0\right\rangle . \tag{IV.2.22}$$

The indices α and β label the multiplets of the SO(3) group. $|0\rangle$ denotes the vacuum state. Because of five different moments of inertia the vectors (IV.2.22) are not the eigenstates of the Hamiltonian (IV.2.21). The action of the Hamiltonian on vectors (IV.2.22) following (IV.2.18) can be expressed in terms of the moments of inertia $a_{(L,M)}$ and the SU(3) group Clebsch-Gordan coefficients.

The chiral symmetry breaking term (III.3.31) for the rational map soliton takes an explicit form like in the $B = 1$ case (III.3.32).

3. Summary

We considered a new rational map approximation ansatz for the Skyrme model which is the noncanonical embedded SU(3) $\supset$ SO(3) soliton with the baryon number $B \geq 2$. The SU(2) rational map ansatz is not spherically symmetric. The canonical quantization leads to five different quantum moments of inertia in the Hamiltonian and the negative quantum mass corrections. The state vectors are defined as the SU(3) group representation (Λ, Θ) matrix depending on eight quantum variables q^i because the ansatz does not commute with any generator of the group. The vectors (IV.2.22) are not the eigenvectors of the Hamiltonian for higher representations. The mixing is small. To find the eigenstate vectors, the Hamiltonian matrix must be diagonalized in every (Λ, Θ) representation. If the baryon number $B = 1$ and $\hat{n} = \hat{x}$, we get a soliton with two different moments of inertia which was considered in previous chapter.

Concluding statements

- The canonically quantized Skyrme model is extended to general irreducible representations (λ, μ) of SU(3), which can be treated as new discrete phenomenological parameters. In the classical case the representation dependence is a common factor in the Lagrangian, while the quantum corrections essentialy depend on the representation in the quantum case.

- The representation dependence of the Wess-Zumino term arises into an factor, which is proportional to the cubic Casimir operator value, with an exception of the self adjoint irreducible representations in when this term vanishes.

- The symmetry breaking term has a diverse functional dependence on the profile function $F(r)$ in different irreducible representations (λ, μ). In the case of self adjoint representations the symmetry breaking term reduces to the SU(2) form.

- The new ansatz for the Skyrme model, which is defined in the noncanonical SU(3) $\supset$ SO(3) bases, is introduced. The canonical quantization of the soliton leads to two moments of inertia one of which coincides with the SU(2) soliton moments of inertia, and new expressions of the quantum mass corrections. For the noncanonically embedded SO(3) soliton the Wess-Zumino term is equal to zero.

- The rational map approximation ansatz for the Skyrme model, of the noncanonically embedded SU(3) $\supset$ SO(3) soliton with the baryon number $B \geq 2$, is investigated. Five different quantum moments of inertia and new quantum mass corrections follow from the canonical quantization. Because of five different moments of inertia the state vectors are not the eigenvectors of the Hamiltonian for higher representations. The explored ansatz can be used to describe light nuclei as special skyrmions.

Appendix A. Definitions for the SU(2) soliton

The Wigner D^j function parametrization in the form [111]

$$D^j_{m,m'}(\alpha,\beta,\gamma) = \langle j,m|e^{-\mathrm{i}\alpha\hat{J}_3}e^{-\mathrm{i}\beta\hat{J}_2}e^{-\mathrm{i}\gamma\hat{J}_3}|j,m'\rangle, \tag{A.1}$$

allows to obtain the following relations:

$$\frac{\partial}{\partial\alpha_i}D^j_{m,n}(\alpha_1,\alpha_2,\alpha_3) = \partial_i D^j_{m,n}(\alpha) = -\frac{1}{\sqrt{2}}\,C_i^{(a)}(\alpha)\Big\langle j,m\Big|J_a\Big|j,m'\Big\rangle D^j_{m',n}(\alpha), \tag{A.2a}$$

$$\partial_i D^j_{m,n}(-\alpha) = \frac{1}{\sqrt{2}}\,C_i^{(a)}(\alpha)\;D^j_{m,n'}(-\alpha)\Big\langle j,n'\Big|J_a\Big|j,n\Big\rangle, \tag{A.2b}$$

$$\partial_i D^j_{m,n}(\alpha) = -\frac{1}{\sqrt{2}}\,C_i^{\prime(a)}(\alpha)\;D^j_{m,m'}(\alpha)\Big\langle j,m'\Big|J_a\Big|j,n\Big\rangle, \tag{A.2c}$$

$$\partial_i D^j_{m,n}(-\alpha) = \frac{1}{\sqrt{2}}\,C_i^{\prime(a)}(\alpha)\Big\langle j,m\Big|J_a\Big|j,n'\Big\rangle D^j_{n'n}(-\alpha). \tag{A.2d}$$

The coefficients $C_i^a(\alpha)$ and the Wigner matrix satisfy the relations

$$C_i^{\prime(a)}(\alpha) = D^1_{a,a'}(-\alpha)\;C_i^{(a')}(\alpha), \tag{A.3a}$$

$$C_i^{(a)}(\alpha) = D^1_{a,a'}(\alpha)\;C_i^{\prime(a')}(\alpha), \tag{A.3b}$$

Their the explicit forms [11] are listed below:

$D^1_{mm'}(\alpha,\beta,\gamma)$	$m'=1$	$m'=0$	$m'=-1$
$m=1$	$\mathrm{e}^{-\mathrm{i}\alpha}\,\frac{(1+\cos\beta)}{2}\,\mathrm{e}^{-\mathrm{i}\gamma}$	$\mathrm{e}^{-\mathrm{i}\alpha}\,\frac{-\sin\beta}{\sqrt{2}}$	$\mathrm{e}^{-\mathrm{i}\alpha}\,\frac{(1-\cos\beta)}{2}\,\mathrm{e}^{\mathrm{i}\gamma}$
$m=0$	$\frac{\sin\beta}{\sqrt{2}}\,\mathrm{e}^{-\mathrm{i}\gamma}$	$\cos\beta$	$\frac{-\sin\beta}{\sqrt{2}}\,\mathrm{e}^{\mathrm{i}\gamma}$
$m=-1$	$\mathrm{e}^{\mathrm{i}\alpha}\,\frac{(1-\cos\beta)}{2}\,\mathrm{e}^{-\mathrm{i}\gamma}$	$\mathrm{e}^{\mathrm{i}\alpha}\,\frac{\sin\beta}{\sqrt{2}}$	$\mathrm{e}^{\mathrm{i}\alpha}\,\frac{(1+\cos\beta)}{2}\,\mathrm{e}^{\mathrm{i}\gamma}$

(A.4)

$C_i^{(a)}(\alpha)$	$i=1$	$i=2$	$i=3$
$a=+$	0	$-\mathrm{e}^{-\mathrm{i}\alpha_1}$	$-\mathrm{i}\sin\alpha_2\mathrm{e}^{-\mathrm{i}\alpha_1}$
$a=0$	$\mathrm{i}\sqrt{2}$	0	$\mathrm{i}\sqrt{2}\cos\alpha_2$
$a=-$	0	$-\mathrm{e}^{\mathrm{i}\alpha_1}$	$\mathrm{i}\sin\alpha_2\mathrm{e}^{\mathrm{i}\alpha_1}$

(A.5)

$C^i_{(a)}(\alpha)$	$a=+$	$a=0$	$a=-$
$i=1$	$-\frac{\mathrm{i}}{2}\cot\alpha_2 \mathrm{e}^{\mathrm{i}\alpha_1}$	$-\frac{\mathrm{i}}{\sqrt{2}}$	$\frac{\mathrm{i}}{2}\cot\alpha_2 \mathrm{e}^{-\mathrm{i}\alpha_1}$
$i=2$	$-\frac{1}{2}\mathrm{e}^{\mathrm{i}\alpha_1}$	0	$-\frac{1}{2}\mathrm{e}^{-\mathrm{i}\alpha_1}$
$i=3$	$\frac{\mathrm{i}}{2}\frac{1}{\sin\alpha_2}\mathrm{e}^{\mathrm{i}\alpha_1}$	0	$-\frac{\mathrm{i}}{2}\frac{1}{\sin\alpha_2}\mathrm{e}^{-\mathrm{i}\alpha_1}$

(A.6)

The coefficients $C^a_i(\alpha)$ also satisfy the orthogonality relations

$$C_i^{(a)}(\alpha)\, C^i_{(b)}(\alpha) = C_i'^{(a)}(\alpha)\, C'^i_{(b)}(\alpha) = \delta_{a,b}, \tag{A.7a}$$

$$C_l^{(a)}(\alpha)\, C^k_{(a)}(\alpha) = C_l'^{(a)}(\alpha)\, C'^k_{(a)}(\alpha) = \delta_{l,k}. \tag{A.7b}$$

The Wigner inverse function D^{-1} can be denoted in several ways:

$$\Big(D^{-1}(\alpha,\beta,\gamma)\Big)^j_{m,m'} = D^{\dagger j}_{m,m'}(\alpha,\beta,\gamma) = D^{*j}_{m',m}(\alpha,\beta,\gamma) = D^j_{m,m'}(-\gamma,-\beta,-\alpha). \tag{A.8}$$

Some differential forms of the SU(2) Wigner matrices do not depend on parametrization (A.1) but preserve orthogonality relations (A.3) and (A.7), and are useful in calculations:

$$\begin{aligned}\Big(\frac{\partial}{\partial x_k}D^j_{m,n}(\alpha)\Big)\, D^j_{n,m'}(-\alpha) &= -\frac{1}{\sqrt{2}}\,\frac{\partial}{\partial x_k}\alpha^i\cdot C_i^{(a)}(\alpha)\big\langle j,m\big|J_a\big|j,m'\big\rangle \\ &= -\sqrt{\tfrac{1}{2}j(j+1)}\,\frac{\partial}{\partial x_k}\alpha^i\cdot C_i^{(a)}(\alpha)\begin{bmatrix} j & 1 & j\\ m' & a & m\end{bmatrix};\end{aligned} \tag{A.9a}$$

$$\begin{aligned}D^j_{m,n}(-\alpha)\Big(\frac{\partial}{\partial x_k}D^j_{n,m'}(\alpha)\Big) &= -\frac{1}{\sqrt{2}}\,\frac{\partial}{\partial x_k}\alpha^i\cdot C_i'^{(a)}(\alpha)\big\langle j,m\big|J_a\big|j,m'\big\rangle \\ &= -\sqrt{\tfrac{1}{2}j(j+1)}\,\frac{\partial}{\partial x_k}\alpha^i\cdot C_i'^{(a)}(\alpha)\begin{bmatrix} j & 1 & j\\ m' & a & m\end{bmatrix};\end{aligned} \tag{A.9b}$$

$$\begin{aligned}\Big(\frac{\partial}{\partial x_k}D^j_{m,n}(\alpha)\Big)\, D^j_{n,m'}(-\alpha) &= \sqrt{\tfrac{1}{2}j(j+1)}\Big(\frac{\partial}{\partial x_k}D^1_{n,n''}(\alpha)\Big)D^1_{n'',n'}(-\alpha)\begin{bmatrix} 1 & 1 & 1\\ n' & a & n\end{bmatrix}\begin{bmatrix} j & 1 & j\\ m' & a & m\end{bmatrix} \\ &= \Big(\frac{\partial}{\partial x_k}D^1_{n,n''}(\alpha)\Big)D^1_{n'',n'}(-\alpha)\begin{bmatrix} 1 & 1 & 1\\ n' & a & n\end{bmatrix}\big\langle j,m\big|J_a\big|j,m'\big\rangle.\end{aligned} \tag{A.9c}$$

The unit vector $\hat{x} = \frac{x}{r}$ in the contravariant circular coordinates is defined in respect to the Cartesian, spherical and circular covariant coordinate systems as

$$x^{+1} = -\frac{1}{\sqrt{2}}(x_1 - \mathrm{i}x_2) = -\frac{1}{\sqrt{2}}\sin\vartheta\,\mathrm{e}^{-\mathrm{i}\varphi} = -x_{-1}, \tag{A.10a}$$

$$x^0 = x_3 = \cos\vartheta = x_0, \tag{A.10b}$$

$$x^{-1} = \frac{1}{\sqrt{2}}(x_1 + \mathrm{i}x_2) = \frac{1}{\sqrt{2}}\sin\vartheta\,\mathrm{e}^{\mathrm{i}\varphi} = -x_{+1}, \tag{A.10c}$$

respectively. The unit vector can also be expressed by spherical harmonics

$$\hat{x}_a = \frac{2\sqrt{\pi}}{\sqrt{3}}Y_{1,a}(\vartheta,\varphi), \tag{A.11}$$

where the angles θ and φ are the polar angles that define the direction of the unit vector $\hat{x}$ in the spherical coordinates.

The Euler angles of the $D(\alpha)$ Wigner matrix are expresed in terms of the profile function $F(r)$

$$\alpha_1(x) = \varphi - \arctan(\cos\vartheta\, \tan F(r)) - \frac{\pi}{2}, \tag{A.12a}$$
$$\alpha_2(x) = -2\arcsin(\sin\vartheta\, \sin F(r)), \tag{A.12b}$$
$$\alpha_3(x) = -\varphi - \arctan(\cos\vartheta\, \tan F(r)) + \frac{\pi}{2}. \tag{A.12c}$$

The differential operators in the spherical coordinates is defined as [111]

$$\frac{\partial}{\partial x} = \sin\vartheta\cos\varphi\,\frac{\partial}{\partial r} + \frac{1}{r}\cos\vartheta\cos\varphi\,\frac{\partial}{\partial\vartheta} - \frac{\sin\varphi}{r\,\sin\vartheta}\,\frac{\partial}{\partial\varphi}, \tag{A.13a}$$
$$\frac{\partial}{\partial y} = \sin\vartheta\sin\varphi\,\frac{\partial}{\partial r} + \frac{1}{r}\cos\vartheta\sin\varphi\,\frac{\partial}{\partial\vartheta} + \frac{\cos\varphi}{r\,\sin\vartheta}\,\frac{\partial}{\partial\varphi}, \tag{A.13b}$$
$$\frac{\partial}{\partial z} = \cos\vartheta\,\frac{\partial}{\partial r} - \frac{1}{r}\sin\vartheta\,\frac{\partial}{\partial\vartheta}. \tag{A.13c}$$

The circular components of the differential operator are:

$$\nabla_{+1} = -\frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\Big) = -\frac{1}{\sqrt{2}}\mathrm{e}^{\mathrm{i}\varphi}\Big(\sin\vartheta\,\frac{\partial}{\partial r} + \frac{1}{r}\cos\vartheta\,\frac{\partial}{\partial\vartheta} + \frac{\mathrm{i}}{r\,\sin\vartheta}\,\frac{\partial}{\partial\varphi}\Big), \tag{A.14a}$$
$$\nabla_0 = \frac{\partial}{\partial z} = \cos\vartheta\,\frac{\partial}{\partial r} - \frac{1}{r}\sin\vartheta\,\frac{\partial}{\partial\vartheta}, \tag{A.14b}$$
$$\nabla_{-1} = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\Big) = \frac{1}{\sqrt{2}}\mathrm{e}^{-\mathrm{i}\varphi}\Big(\sin\vartheta\,\frac{\partial}{\partial r} + \frac{1}{r}\cos\vartheta\,\frac{\partial}{\partial\vartheta} - \frac{\mathrm{i}}{r\,\sin\vartheta}\,\frac{\partial}{\partial\varphi}\Big). \tag{A.14c}$$

The actions of the differential operator are:

$$\begin{gathered}\nabla_b\, r = \hat{x}_b, \qquad \nabla_b\, F(r) = \hat{x}_b\, F'(r), \qquad \nabla_b\, \hat{x}_a = \frac{1}{r}\big((-1)^b\,\delta_{-b,a} - \hat{x}_b\,\hat{x}_a\big),\\ \nabla_b\, x^a = \delta_{b,a}, \qquad \nabla_b\, x_a = (-1)^b\,\delta_{-b,a}.\end{gathered} \tag{A.15}$$

In the rational map case it is useful to use the modified differential operator $\bar{\nabla}_a$, which is defined in coordinate system with the unit vectors $\hat{n}_a$ of the rational map representations:

$$\begin{gathered}\nabla_a = b_{a,a'}\bar{\nabla}_{a'}, \qquad \bar{\nabla}_{a'}(r\,\hat{n}^a) = \delta_{a,a'}, \qquad b_{a,a'} = \nabla_a\big(r\,\hat{n}^{a'}\big) = (-1)^{a'}\nabla_a\big(r\,\hat{n}_{-a'}\big),\\ (-1)^s\, b_{-s,m}\, b_{s,m'} = (-1)^s(\nabla_{-s} r\,\hat{n}_m)(\nabla_s r\,\hat{n}_{m'}) = \hat{n}_m\hat{n}_{m'} + \mathcal{I}\big((-1)^m\delta_{-m,m'} - \hat{n}_m\hat{n}_{m'}\big).\end{gathered} \tag{A.16}$$

The SU(2) Wigner matrix expressions in the circular coordinates:

$$\begin{aligned} D^1_{a,a'}(\alpha) &= 2\sin^2 F\,(-1)^a\hat{x}_{-a}\hat{x}_{a'} & D^1_{a,a'}(-\alpha) &= 2\sin^2 F\,(-1)^a\hat{x}_{-a}\hat{x}_{a'}\\ &+ \mathrm{i}\sqrt{2}\sin 2F \begin{bmatrix} 1 & 1 & 1\\ a & u & a'\end{bmatrix}\hat{x}_u; & &- \mathrm{i}\sqrt{2}\sin 2F \begin{bmatrix} 1 & 1 & 1\\ a & u & a'\end{bmatrix}\hat{x}_u\\ &+ \cos 2F\,\delta_{a,a'} & &+ \cos 2F\,\delta_{a,a'}.\end{aligned} \tag{A.17}$$

Here the inverses matrices $D(-\alpha)$ are obtained from $D(\alpha)$ by changing the sign of F.

The differential forms of the SU(2) Wigner matrices

$$\begin{aligned}\nabla_b D^1_{a,a'}(\alpha) = & \, 2\left(\sin 2F \cdot F' - \frac{2}{r}\sin^2 F\right)(-1)^a \hat{x}_b \hat{x}_{-a}\hat{x}_{a'} \\ & + \mathrm{i}\sqrt{2}\left(2\cos 2F \cdot F' - \frac{1}{r}\sin 2F\right)\begin{bmatrix} 1 & 1 & 1 \\ a & u & a' \end{bmatrix}\hat{x}_u \hat{x}_b \\ & + \frac{2}{r}\sin^2 F\big(\delta_{b,a}\hat{x}_{a'} + (-1)^{a+b}\delta_{-b,a'}\hat{x}_{-a}\big) - 2\sin 2F \cdot F' \, \delta_{a,a'}\hat{x}_b \\ & + \mathrm{i}\sqrt{2}\,\frac{1}{r}\,\sin 2F\begin{bmatrix} 1 & 1 & 1 \\ a' & b & a \end{bmatrix},\end{aligned} \tag{A.18a}$$

and their combinations of two, three and four Wigner matrices:

$$\begin{aligned}\big(\nabla_b D^1_{a,a''}(\alpha)\big)\, D^1_{a'',a'}(-\alpha) = & \, \mathrm{i}\sqrt{2}\left(2F' - \frac{1}{r}\sin 2F\right)\begin{bmatrix} 1 & 1 & 1 \\ a & u & a' \end{bmatrix}\hat{x}_u \hat{x}_b \\ & + \frac{2}{r}\sin^2 F\left(\delta_{a,b}\,\hat{x}_{a'} - (-1)^{a+b}\delta_{b,-a'}\,\hat{x}_{-a}\right) \\ & + \mathrm{i}\,\frac{\sqrt{2}}{r}\,\sin 2F\begin{bmatrix} 1 & 1 & 1 \\ a' & b & a \end{bmatrix},\end{aligned} \tag{A.18b}$$

$$\begin{aligned} & (-1)^b\Big(\nabla_b D^1_{a,a_1}(\alpha)\Big) D^1_{a_1,a_2}(-\alpha)\Big(\nabla_{-b} D^1_{a_2,a'}(\alpha)\Big) = \\ & \quad = 4\Big(F'^2\cos 2F - \frac{1}{r^2}\sin^2 F(1 + 2\sin^2 F)\Big)(-1)^a \hat{x}_{-a}\hat{x}_{a'} \\ & \quad - \mathrm{i}\,4\sqrt{2}\,\sin 2F\Big(F'^2 + \frac{1}{r^2}\sin^2 F\Big)\begin{bmatrix} 1 & 1 & 1 \\ a & u & a' \end{bmatrix}\hat{x}_u \\ & \quad - 4\cos 2F\Big(F'^2 + \frac{1}{r^2}\sin^2 F\Big)\delta_{a,a'},\end{aligned} \tag{A.18c}$$

$$\begin{aligned} & (-1)^b D^1_{a,a_1}(-\alpha)\Big(\nabla_b D^1_{a_1,a_2}(\alpha)\Big) D^1_{a_2,a_3}(-\alpha)\Big(\nabla_{-b} D^1_{a_3,a'}(\alpha)\Big) = \\ & (-1)^b\Big(\nabla_b D^1_{a,a_1}(\alpha)\Big) D^1_{a_1,a_2}(-\alpha)\Big(\nabla_{-b} D^1_{a_2,a_3}(\alpha)\Big) D^1_{a_3,a'}(-\alpha) = \\ & \qquad\qquad = \; 4\Big(F'^2 - \frac{1}{r^2}\sin^2 F\Big)(-1)^a \hat{x}_{-a}\hat{x}_{a'} \\ & \qquad\qquad - 4\Big(F'^2 + \frac{1}{r^2}\sin^2 F\Big)\delta_{a,a'}.\end{aligned} \tag{A.18d}$$

Integrals of the Wigner D matrices:

$$\begin{aligned} & \int_0^{2\pi}\!\!\int_0^{\pi} D^{I_1}_{m_1,m'_1}(\alpha) D^{I_2}_{m_2,m'_2}(\alpha)\sin\theta\,\mathrm{d}\theta\mathrm{d}\varphi = \\ & \quad = \sum_{I=|I_1-I_2|}^{I_1+I_2}\sum_{m,m'}\begin{bmatrix} I_1 & I_2 & I \\ m_1 & m_2 & m \end{bmatrix}\begin{bmatrix} I_1 & I_2 & I \\ m'_1 & m'_2 & m' \end{bmatrix}\int_0^{2\pi}\!\!\int_0^{\pi} D^I_{m,m'}(\alpha)\sin\theta\,\mathrm{d}\theta\mathrm{d}\varphi \\ & \quad = \sum_{I=|I_1-I_2|}^{I_1+I_2}\sum_{m}\begin{bmatrix} I_1 & I_2 & I \\ m_1 & m_2 & m \end{bmatrix}\begin{bmatrix} I_1 & I_2 & I \\ m'_1 & m'_2 & m \end{bmatrix} d_I,\end{aligned} \tag{A.19}$$

where d_I:

$$
\begin{aligned}
d_0 &= 4\pi; \\
d_{\frac{1}{2}} &= 4\pi \cos F; \\
d_1 &= 4\pi(1 - \frac{4}{3}\sin^2 F); \\
d_{\frac{3}{2}} &= 4\pi \cos F(1 - 2\sin^2 F); \\
d_2 &= 4\pi(1 - \frac{4}{5}(\sin^2 F + \sin^2 2F)).
\end{aligned} \tag{A.20}
$$

The SU(2) $C(\alpha)$ matrices can be expressed in terms of the Wigner matrices:

$$
\partial_k \alpha^i C_i^{(a)}(\alpha) = -\frac{3\sqrt{2}}{\sqrt{j(j+1)(2j+1)^2}} \begin{bmatrix} j & 1 & j \\ m' & a & m \end{bmatrix} \partial_k D^j_{m,n}(\alpha) D^j_{n,m'}(-\alpha); \tag{A.21a}
$$

$$
\partial_k \alpha^i C_i'^{(a)}(\alpha) = -\frac{3\sqrt{2}}{\sqrt{j(j+1)(2j+1)^2}} \begin{bmatrix} j & 1 & j \\ m' & a & m \end{bmatrix} D^j_{m,n}(-\alpha) \partial_k D^j_{n,m'}(\alpha). \tag{A.21b}
$$

Expressions of the SU(2) $C(\alpha)$ matrices in terms of the functions F and the vectors $\hat{x}$:

$$
\begin{aligned}
-\sqrt{2}\nabla_b \alpha^i C_i'^{(0,1,a)}(\alpha) = \nabla_b \alpha^i C_i^{(a)}(\alpha) &= -\mathrm{i}2\sqrt{2}\left(F' - \frac{1}{2r}\sin 2F\right)(-1)^a \hat{x}_{-a}\hat{x}_b \\
&+ \frac{4}{r}\sin^2 F \begin{bmatrix} 1 & 1 & 1 \\ u & a & b \end{bmatrix} \hat{x}_u - \mathrm{i}\frac{\sqrt{2}}{r}\sin 2F\,\delta_{a,b};
\end{aligned} \tag{A.22a}
$$

$$
\begin{aligned}
-\sqrt{2}\nabla_b \alpha^i C_i^{(0,1,a)}(\alpha) = \nabla_b \alpha^i C_i'^{(a)}(\alpha) &= -\mathrm{i}2\sqrt{2}\left(F' - \frac{1}{2r}\sin 2F\right)(-1)^a \hat{x}_{-a}\hat{x}_b \\
&- \frac{4}{r}\sin^2 F \begin{bmatrix} 1 & 1 & 1 \\ u & a & b \end{bmatrix} \hat{x}_u - \mathrm{i}\frac{\sqrt{2}}{r}\sin 2F\,\delta_{a,b}.
\end{aligned} \tag{A.22b}
$$

Some combinations of the $C(\alpha)$ matrices used in calculations:

$$
\begin{aligned}
&(-1)^b\,\nabla_b \alpha^i C_i^{(d)}(\alpha) \nabla_{-b} \alpha^{i'} C_{i'}^{(d')}(\alpha) = \\
&\quad = 8\left(\frac{1}{r^2}\sin^2 F - F'^2\right)(-1)^{d+d'}\hat{x}_{-d}\hat{x}_{-d'} - \frac{8}{r^2}\sin^2 F\,(-1)^d\,\delta_{d,-d'};
\end{aligned} \tag{A.23a}
$$

$$
\begin{aligned}
&(-1)^b\,\nabla_b \alpha^i C_i^{(d)}(\alpha) \nabla_{-b} \alpha^{i'} C_{i'}'^{(d')}(\alpha) = \\
&\quad = 8\left(\frac{1}{r^2}\cos 2F \sin^2 F - F'^2\right)(-1)^{d+d'}\hat{x}_{-d}\hat{x}_{-d'} - \frac{8}{r^2}\cos 2F \sin^2 F\,(-1)^d\,\delta_{d,-d'} \\
&\quad + \mathrm{i}\frac{16\sqrt{2}}{r^2}\cos F \sin^3 F\,(-1)^{d'} \begin{bmatrix} 1 & 1 & 1 \\ u & d & -d' \end{bmatrix} \hat{x}_u;
\end{aligned} \tag{A.23b}
$$

$$
\begin{aligned}
&(-1)^b\,\nabla_b \alpha^i C_i'^{(d)}(\alpha) \nabla_{-b} \alpha^{i'} C_{i'}^{(d')}(\alpha) = \\
&\quad = 8\left(\frac{1}{r^2}\cos 2F \sin^2 F - F'^2\right)(-1)^{d+d'}\hat{x}_{-d}\hat{x}_{-d'} - \frac{8}{r^2}\cos 2F \sin^2 F\,(-1)^{d'}\,\delta_{d',-d} \\
&\quad + \mathrm{i}\frac{16\sqrt{2}}{r^2}\cos F \sin^3 F\,(-1)^{d} \begin{bmatrix} 1 & 1 & 1 \\ u & d' & -d \end{bmatrix} \hat{x}_u.
\end{aligned} \tag{A.23c}
$$

Appendix B. Definitions for the SU(3) soliton

The functions $C'^{(\bar{A})}_{\alpha}(q)$ defined in (II.1.2) constitute nonsingular 7×7 matrices. We can introduce the reciprocal functions $C'^{\alpha}_{(\bar{B})}(q)$ by

$$\sum_{\bar{A}} C'^{(\bar{A})}_{\alpha}(q)\cdot C'^{\beta}_{(\bar{A})}(q)=\delta_{\alpha,\beta}\,, \tag{B.1a}$$

$$\sum_{\alpha} C'^{(\bar{A})}_{\alpha}(q)\cdot C'^{\alpha}_{(\bar{B})}(q)=\delta_{(\bar{A})(\bar{B})}\,. \tag{B.1b}$$

Here $(\bar{A})$ and $(\bar{B})$ denote the basis of the irrep $(1,1)$, with an exception of the state $(0,0,0)$. The functions $C'^{\alpha}_{(0)}(q)$ are not defined.

The properties of the functions $C'^{(K)}_{\alpha}(q)$ follow from $\partial_\alpha\partial_\beta D^{(\lambda,\mu)}=\partial_\beta\partial_\alpha D^{(\lambda,\mu)}$:

$$\partial_\beta C'^{(K)}_{\alpha}(q)-\partial_\alpha C'^{(K)}_{\beta}(q)-\sqrt{3}\begin{bmatrix}(1,1) & (1,1) & (1,1)_a\\ (K') & (K'') & (K)\end{bmatrix} C'^{(K')}_{\beta}(q)C'^{(K'')}_{\alpha}(q)=0\,, \tag{B.2}$$

and are correct for all states (K) including $(0,0,0)$. The following properties of the functions $C'^{\alpha}_{(\bar{K})}(q)$ are useful:

$$\begin{aligned}&C'^{\alpha}_{(\bar{K}')}(q)\partial_\alpha C'^{\beta}_{(\bar{K}'')}(q)-C'^{\alpha}_{(\bar{K}'')}(q)\partial_\alpha C'^{\beta}_{(\bar{K}')}(q)+\sqrt{3}\begin{bmatrix}(1,1) & (1,1) & (1,1)_a\\ (\bar{K}') & (\bar{K}'') & (\bar{K})\end{bmatrix} C'^{\beta}_{(\bar{K})}(q)\\ &=\sqrt{3}z''C'^{(0)}_{\alpha}(q)C'^{\alpha}_{(\bar{K}')}(q)C'^{\beta}_{(\bar{K}'')}(q)-\sqrt{3}z'C'^{(0)}_{\alpha}(q)C'^{\alpha}_{(\bar{K}'')}(q)C'^{\beta}_{(\bar{K}')}(q).\end{aligned} \tag{B.3}$$

The second order term of the Skyrme Lagrangian (I.3.21)

$$\begin{aligned}\mathrm{Tr}\Big(\dot{U}\overset{+}{U}\dot{U}\overset{+}{U}\Big)=&\,\mathrm{Tr}\Big(A\overset{+}{A}\dot{A}\overset{+}{A}\dot{A}\overset{+}{A}\Big)+\mathrm{Tr}\Big(AU_0\overset{+}{A}\dot{A}\overset{+}{A}\dot{A}\overset{+}{U}_0\overset{+}{A}\Big)\\ &-\mathrm{Tr}\Big(A\overset{+}{A}\dot{A}U_0\overset{+}{A}\dot{A}\overset{+}{U}_0\overset{+}{A}\Big)-\mathrm{Tr}\Big(AU_0\overset{+}{A}\dot{A}\overset{+}{U}_0\overset{+}{A}\dot{A}\overset{+}{A}\Big),\end{aligned} \tag{B.4}$$

and the fourth order term

$$\begin{aligned}\mathrm{Tr}\bigg(\Big[\dot{U}\overset{+}{U},(\partial_k U)\overset{+}{U}\Big]\Big[\dot{U}\overset{+}{U},(\partial^k U)\,\overset{+}{U}\Big]\bigg)&=-(-1)^c\,\mathrm{Tr}\bigg(\Big[\dot{U}\overset{+}{U},(\nabla_c U)\,\overset{+}{U}\Big]\Big[\dot{U}\overset{+}{U},(\nabla_{-c} U)\,\overset{+}{U}\Big]\bigg)\\ =&-(-1)^c\,\mathrm{Tr}\bigg(A\Big[\overset{+}{A}\dot{A},(\nabla_c U_0)\,\overset{+}{U}_0\Big]\Big[\overset{+}{A}\dot{A},(\nabla_{-c} U_0)\,\overset{+}{U}_0\Big]\overset{+}{A}\bigg)\\ &-(-1)^c\,\mathrm{Tr}\bigg(AU_0\Big[\overset{+}{A}\dot{A},\overset{+}{U}_0\,(\nabla_c U_0)\Big]\Big[\overset{+}{A}\dot{A},\overset{+}{U}_0\,(\nabla_{-c} U_0)\Big]\overset{+}{U}_0\overset{+}{A}\bigg)\\ &+(-1)^c\,\mathrm{Tr}\bigg(A\Big[\overset{+}{A}\dot{A},(\nabla_c U_0)\,\overset{+}{U}_0\Big]U_0\Big[\overset{+}{A}\dot{A},\overset{+}{U}_0\,(\nabla_{-c} U_0)\Big]\overset{+}{U}_0\overset{+}{A}\bigg)\\ &+(-1)^c\,\mathrm{Tr}\bigg(AU_0\Big[\overset{+}{A}\dot{A},\overset{+}{U}_0\,(\nabla_c U_0)\Big]\overset{+}{U}_0\Big[\overset{+}{A}\dot{A},(\nabla_{-c} U_0)\,\overset{+}{U}_0\Big]\overset{+}{A}\bigg),\end{aligned} \tag{B.5}$$

are expressed in more usable form for quantization procedure.

In the case of the fundamental representation the Λ_k matrix generators (II.1.3) reduce to the standard Gell–Mann matrices λ_i:

$$
\left\langle (1,0)\right| J^{(1,1)}_{0,0,0} \left|(1,0)\right\rangle = \begin{array}{c|c} & \left|z{=}-\tfrac{1}{2},j{=}\tfrac{1}{2},m{=}\tfrac{1}{2}\right\rangle \quad \left|-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2}\right\rangle \quad \left|0,0,0\right\rangle \\ \hline \begin{array}{l} \left\langle z{=}-\tfrac{1}{2},j{=}\tfrac{1}{2},m{=}\tfrac{1}{2}\right| \\ \left\langle -\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2}\right| \\ \left\langle 0,0,0\right| \end{array} & \begin{pmatrix} -\dfrac{1}{2\sqrt{3}} & 0 & 0 \\ 0 & -\dfrac{1}{2\sqrt{3}} & 0 \\ 0 & 0 & \dfrac{1}{\sqrt{3}} \end{pmatrix} \end{array}
$$

$$
\left\langle (1,0)\right| J^{(1,1)}_{0,1,0} \left|(1,0)\right\rangle = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

$$
\left\langle (1,0)\right| J^{(1,1)}_{0,1,1} \left|(1,0)\right\rangle = \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \left\langle (1,0)\right| J^{(1,1)}_{0,1,-1} \left|(1,0)\right\rangle = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

$$
\left\langle (1,0)\right| J^{(1,1)}_{-\frac{1}{2},\frac{1}{2},\frac{1}{2}} \left|(1,0)\right\rangle = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \left\langle (1,0)\right| J^{(1,1)}_{-\frac{1}{2},\frac{1}{2},-\frac{1}{2}} \left|(1,0)\right\rangle = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \end{pmatrix}
$$

$$
\left\langle (1,0)\right| J^{(1,1)}_{\frac{1}{2},\frac{1}{2},\frac{1}{2}} \left|(1,0)\right\rangle = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \qquad \left\langle (1,0)\right| J^{(1,1)}_{\frac{1}{2},\frac{1}{2},-\frac{1}{2}} \left|(1,0)\right\rangle = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix} \tag{B.6}
$$

Symbols by the first matrix denote the basis functions $\left\langle z, j, m\right|$.

The antisymmetrical isoscalar factors are:

$$
\begin{aligned}
&\begin{bmatrix} (1,1) & (1,1) & (1,1)_{\gamma=1} \\ {}_{(Y)}Z,I & {}_{(1)}-\frac{1}{2},\frac{1}{2} & {}_{(Y+1)}Z-\frac{1}{2},I+\frac{1}{2} \end{bmatrix} = \left[\frac{(I-Z+1)(Z-I+1)(I-Z+3)}{3{\cdot}4(I+1)}\right]^{\frac{1}{2}};\\
&\begin{bmatrix} (1,1) & (1,1) & (1,1)_{\gamma=1} \\ {}_{(Y)}Z,I & {}_{(1)}-\frac{1}{2},\frac{1}{2} & {}_{(Y+1)}Z-\frac{1}{2},I-\frac{1}{2} \end{bmatrix} = \left[\frac{(I+Z)(I+Z+2)(-I-Z+2)}{3{\cdot}4{\cdot}I}\right]^{\frac{1}{2}};\\
&\begin{bmatrix} (1,1) & (1,1) & (1,1)_{\gamma=1} \\ {}_{(Y)}Z,I & {}_{(-1)}\frac{1}{2},\frac{1}{2} & {}_{(Y-1)}Z+\frac{1}{2},I+\frac{1}{2} \end{bmatrix} = -\left[\frac{(I+Z+1)(I+Z+3)(-I-Z+1)}{3{\cdot}4(I+1)}\right]^{\frac{1}{2}};\\
&\begin{bmatrix} (1,1) & (1,1) & (1,1)_{\gamma=1} \\ {}_{(Y)}Z,I & {}_{(-1)}\frac{1}{2},\frac{1}{2} & {}_{(Y-1)}Z+\frac{1}{2},I-\frac{1}{2} \end{bmatrix} = \left[\frac{(I-Z)(I-Z+2)(Z-I+2)}{3{\cdot}4{\cdot}I}\right]^{\frac{1}{2}};\\
&\begin{bmatrix} (1,1) & (1,1) & (1,1)_{\gamma=1} \\ {}_{(Y)}Z,I & {}_{(0)}0,1 & {}_{(Y)}Z,I\pm1 \end{bmatrix} = 0;\\
&\begin{bmatrix} (1,1) & (1,1) & (1,1)_{\gamma=1} \\ {}_{(Y)}Z,I & {}_{(0)}0,1 & {}_{(Y)}Z,I \end{bmatrix} = \left[\frac{I(I+1)}{3}\right]^{\frac{1}{2}};\\
&\begin{bmatrix} (1,1) & (1,1) & (1,1)_{\gamma=1} \\ {}_{(Y)}Z,I & {}_{(0)}0,0 & {}_{(Y)}Z,I \end{bmatrix} = Z.
\end{aligned} \tag{B.7}
$$

The symmetrical isoscalar factors are:

$$\begin{bmatrix} (1,1) & (1,1) & (1,1)_{\gamma=2} \\ {\scriptstyle(Y)}Z,I & {\scriptstyle(1)}-\frac{1}{2},\frac{1}{2} & {\scriptstyle(Y+1)}Z-\frac{1}{2},I+\frac{1}{2} \end{bmatrix} = -\left[\frac{(I-Z+1)(Z-I+1)(I-Z+3)}{3{\cdot}5(I+1)}\right]^{\frac{1}{2}} \times (I+Z+\tfrac{1}{2});$$

$$\begin{bmatrix} (1,1) & (1,1) & (1,1)_{\gamma=2} \\ {\scriptstyle(Y)}Z,I & {\scriptstyle(1)}-\frac{1}{2},\frac{1}{2} & {\scriptstyle(Y+1)}Z-\frac{1}{2},I-\frac{1}{2} \end{bmatrix} = \left[\frac{(I+Z)(I+Z+2)(-I-Z+2)}{3{\cdot}5{\cdot}I}\right]^{\frac{1}{2}} \times (I-Z+\tfrac{1}{2});$$

$$\begin{bmatrix} (1,1) & (1,1) & (1,1)_{\gamma=2} \\ {\scriptstyle(Y)}Z,I & {\scriptstyle(-1)}\frac{1}{2},\frac{1}{2} & {\scriptstyle(Y-1)}Z+\frac{1}{2},I+\frac{1}{2} \end{bmatrix} = -\left[\frac{(I+Z+1)(I+Z+3)(-I-Z+1)}{3{\cdot}5(I+1)}\right]^{\frac{1}{2}} \times (I-Z+\tfrac{1}{2});$$

$$\begin{bmatrix} (1,1) & (1,1) & (1,1)_{\gamma=2} \\ {\scriptstyle(Y)}Z,I & {\scriptstyle(-1)}\frac{1}{2},\frac{1}{2} & {\scriptstyle(Y-1)}Z+\frac{1}{2},I-\frac{1}{2} \end{bmatrix} = -\left[\frac{(I-Z)(I-Z+2)(Z-I+2)}{3{\cdot}5{\cdot}I}\right]^{\frac{1}{2}} \times (I+Z+\tfrac{1}{2});$$

$$\begin{bmatrix} (1,1) & (1,1) & (1,1)_{\gamma=2} \\ {\scriptstyle(Y)}Z,I & {\scriptstyle(0)}0,1 & {\scriptstyle(Y)}Z,I+1 \end{bmatrix} = \left[\frac{(I+Z+1)(I+Z-1)(I-Z+1)(I-Z-1)(I+Z+3)(I-Z+3)}{3{\cdot}5(I+1)(2I+3)}\right]^{\frac{1}{2}};$$

$$\begin{bmatrix} (1,1) & (1,1) & (1,1)_{\gamma=2} \\ {\scriptstyle(Y)}Z,I & {\scriptstyle(0)}0,1 & {\scriptstyle(Y)}Z,I-1 \end{bmatrix} = -\left[\frac{(I+Z)(I-Z)(I+Z+2)(I+Z-2)(I-Z+2)(I-Z-2)}{3{\cdot}5{\cdot}I(2I-1)}\right]^{\frac{1}{2}};$$

$$\begin{bmatrix} (1,1) & (1,1) & (1,1)_{\gamma=2} \\ {\scriptstyle(Y)}Z,I & {\scriptstyle(0)}0,1 & {\scriptstyle(Y)}Z,I \end{bmatrix} = \left[3{\cdot}5{\cdot}I(I+1)\right]^{-\frac{1}{2}} \times \left(Z(I^2+I-Z^2+4)\right);$$

$$\begin{bmatrix} (1,1) & (1,1) & (1,1)_{\gamma=2} \\ {\scriptstyle(Y)}Z,I & {\scriptstyle(0)}0,0 & {\scriptstyle(Y)}Z,I \end{bmatrix} = 5^{-\frac{1}{2}} \times (I^2+I-Z^2-1). \tag{B.8}$$

The isoscalar factors and the Clebsch–Gordan coefficients for the representation (0,0) are [112]:

$$\begin{bmatrix} (\lambda,\mu) & (\mu,\lambda) & (0,0) \\ Z,I & Z',I' & 0,0 \end{bmatrix} = -(-1)^{Z+I}\left[\frac{2(2I+1)}{(\lambda+1)(\mu+1)(\lambda+\mu+2)}\right]^{\frac{1}{2}} \delta_{-Z,Z'}\delta_{I,I'}; \tag{B.9a}$$

$$\begin{bmatrix} (\lambda,\mu) & (\mu,\lambda) & (0,0) \\ Z,I,M & Z',I',M' & 0,0,0 \end{bmatrix} = -(-1)^{Z+M}\left[\frac{2}{(\lambda+1)(\mu+1)(\lambda+\mu+2)}\right]^{\frac{1}{2}} \delta_{-Z,Z'}\delta_{I,I'}\delta_{-M,M'}. \tag{B.9b}$$

For easier manipulation with expressions we group some symmetry properties of the isoscalar factors

$$\begin{aligned}
&\begin{bmatrix} (1,1) & (1,1) & (1,1)_\gamma \\ {\scriptstyle(Y')}Z',I' & {\scriptstyle(Y'')}Z'',I'' & {\scriptstyle(Y)}Z,I \end{bmatrix} = \\
&= (-1)^{I'+I''-I+\gamma} \begin{bmatrix} (1,1) & (1,1) & (1,1)_\gamma \\ {\scriptstyle(Y'')}Z'',I'' & {\scriptstyle(Y')}Z',I' & {\scriptstyle(Y)}Z,I \end{bmatrix} = \\
&= (-1)^{I'+I''-I+\gamma} \begin{bmatrix} (1,1) & (1,1) & (1,1)_\gamma \\ {\scriptstyle(-Y')}-Z',I' & {\scriptstyle(-Y'')}-Z'',I'' & {\scriptstyle(-Y)}-Z,I \end{bmatrix} = \\
&= (-1)^{I''-I+Z'} \left[\frac{2I''+1}{2I+1}\right]^{\frac{1}{2}} \begin{bmatrix} (1,1) & (1,1) & (1,1)_\gamma \\ {\scriptstyle(-Y')}-Z',I' & {\scriptstyle(Y)}Z,I & {\scriptstyle(Y'')}Z'',I'' \end{bmatrix} = \\
&= (-1)^{I'+Z'+\gamma} \left[\frac{2I''+1}{2I+1}\right]^{\frac{1}{2}} \begin{bmatrix} (1,1) & (1,1) & (1,1)_\gamma \\ {\scriptstyle(Y')}Z',I' & {\scriptstyle(-Y)}-Z,I & {\scriptstyle(-Y'')}-Z'',I'' \end{bmatrix} = \\
&= (-1)^{I'-I-Z''} \left[\frac{2I'+1}{2I+1}\right]^{\frac{1}{2}} \begin{bmatrix} (1,1) & (1,1) & (1,1)_\gamma \\ {\scriptstyle(Y)}Z,I & {\scriptstyle(-Y'')}-Z'',I'' & {\scriptstyle(Y')}Z',I' \end{bmatrix} = \\
&= (-1)^{I''-Z''+\gamma} \left[\frac{2I'+1}{2I+1}\right]^{\frac{1}{2}} \begin{bmatrix} (1,1) & (1,1) & (1,1)_\gamma \\ {\scriptstyle(-Y)}-Z,I & {\scriptstyle(Y'')}Z'',I'' & {\scriptstyle(-Y')}-Z',I' \end{bmatrix},
\end{aligned} \tag{B.10}$$

and the SU(3) Clebsch–Gordan coefficients

$$
\begin{aligned}
&\begin{bmatrix} {\scriptstyle (1,1)} & {\scriptstyle (1,1)} & {\scriptstyle (1,1)_\gamma} \\ {\scriptstyle Z',I',M'} & {\scriptstyle Z'',I'',M''} & {\scriptstyle Z,I,M} \end{bmatrix} = \\
&= (-1)^{\gamma} \begin{bmatrix} {\scriptstyle (1,1)} & {\scriptstyle (1,1)} & {\scriptstyle (1,1)_\gamma} \\ {\scriptstyle Z'',I'',M''} & {\scriptstyle Z',I',M'} & {\scriptstyle Z,I,M} \end{bmatrix} = \\
&\qquad (-1)^{\gamma} \begin{bmatrix} {\scriptstyle (1,1)} & {\scriptstyle (1,1)} & {\scriptstyle (1,1)_\gamma} \\ {\scriptstyle -Z',I',-M'} & {\scriptstyle -Z'',I'',-M''} & {\scriptstyle -Z,I,-M} \end{bmatrix} = \\
&= (-1)^{Z'+M'} \begin{bmatrix} {\scriptstyle (1,1)} & {\scriptstyle (1,1)} & {\scriptstyle (1,1)_\gamma} \\ {\scriptstyle -Z',I',-M'} & {\scriptstyle Z,I,M} & {\scriptstyle Z'',I'',M''} \end{bmatrix} = \\
&\qquad (-1)^{Z'+M'+\gamma} \begin{bmatrix} {\scriptstyle (1,1)} & {\scriptstyle (1,1)} & {\scriptstyle (1,1)_\gamma} \\ {\scriptstyle Z',I',M'} & {\scriptstyle -Z,I,-M} & {\scriptstyle -Z'',I'',-M''} \end{bmatrix} = \\
&= (-1)^{Z''+M''} \begin{bmatrix} {\scriptstyle (1,1)} & {\scriptstyle (1,1)} & {\scriptstyle (1,1)_\gamma} \\ {\scriptstyle Z,I,M} & {\scriptstyle -Z'',I'',-M''} & {\scriptstyle Z',I',M'} \end{bmatrix} = \\
&\qquad (-1)^{Z''+M''+\gamma} \begin{bmatrix} {\scriptstyle (1,1)} & {\scriptstyle (1,1)} & {\scriptstyle (1,1)_\gamma} \\ {\scriptstyle -Z,I,-M} & {\scriptstyle Z'',I'',M''} & {\scriptstyle -Z',I',-M'} \end{bmatrix}.
\end{aligned}
\tag{B.11}
$$

The series of the Clebsch–Gordan coefficients

$$
\begin{aligned}
J^{(1,1)}_{(A)} J^{(1,1)}_{(B)} = &\begin{bmatrix} {\scriptstyle (1,1)} & {\scriptstyle (1,1)} & {\scriptstyle (0,0)} \\ {\scriptstyle (A)} & {\scriptstyle (B)} & {\scriptstyle (0)} \end{bmatrix} J^{(0,0)}_{(0)} + \begin{bmatrix} {\scriptstyle (1,1)} & {\scriptstyle (1,1)} & {\scriptstyle (1,1)_{\gamma=1}} \\ {\scriptstyle (A)} & {\scriptstyle (B)} & {\scriptstyle (C)} \end{bmatrix} J'^{(1,1)}_{(C)} \\
&+ \begin{bmatrix} {\scriptstyle (1,1)} & {\scriptstyle (1,1)} & {\scriptstyle (1,1)_{\gamma=2}} \\ {\scriptstyle (A)} & {\scriptstyle (B)} & {\scriptstyle (C)} \end{bmatrix} J''^{(1,1)}_{(C)} + \begin{bmatrix} {\scriptstyle (1,1)} & {\scriptstyle (1,1)} & {\scriptstyle (3,0)} \\ {\scriptstyle (A)} & {\scriptstyle (B)} & {\scriptstyle (C)} \end{bmatrix} J^{(3,0)}_{(C)} \\
&+ \begin{bmatrix} {\scriptstyle (1,1)} & {\scriptstyle (1,1)} & {\scriptstyle (0,3)} \\ {\scriptstyle (A)} & {\scriptstyle (B)} & {\scriptstyle (C)} \end{bmatrix} J^{(0,3)}_{(C)} + \begin{bmatrix} {\scriptstyle (1,1)} & {\scriptstyle (1,1)} & {\scriptstyle (2,2)} \\ {\scriptstyle (A)} & {\scriptstyle (B)} & {\scriptstyle (C)} \end{bmatrix} J^{(2,2)}_{(C)}.
\end{aligned}
\tag{B.12}
$$

The submatrix elements are expressed by using a series of the Clebsch–Gordan coefficients (B.12)

$$
\begin{aligned}
&\begin{bmatrix} {\scriptstyle (1,1)} & {\scriptstyle (1,1)} & {\scriptstyle (0,0)} \\ {\scriptstyle (A)} & {\scriptstyle (B)} & {\scriptstyle (0)} \end{bmatrix} \left\langle \begin{smallmatrix} (\lambda,\mu) \\ (D) \end{smallmatrix} \middle| J^{(1,1)}_{(A)} J^{(1,1)}_{(B)} \middle| \begin{smallmatrix} (\lambda,\mu) \\ (D) \end{smallmatrix} \right\rangle = \left\langle \begin{smallmatrix} (\lambda,\mu) \\ (D) \end{smallmatrix} \middle| J^{(0,0)}_{(0)} \middle| \begin{smallmatrix} (\lambda,\mu) \\ (D) \end{smallmatrix} \right\rangle \\
&\qquad = \left\langle (\lambda,\mu) \middle\| J^{(0,0)} \middle\| (\lambda,\mu) \right\rangle \begin{bmatrix} {\scriptstyle (\lambda,\mu)} & {\scriptstyle (0,0)} & {\scriptstyle (\lambda,\mu)} \\ {\scriptstyle (D)} & {\scriptstyle (0)} & {\scriptstyle (D)} \end{bmatrix};
\end{aligned}
\tag{B.13a}
$$

$$
\begin{aligned}
&\begin{bmatrix} {\scriptstyle (1,1)} & {\scriptstyle (1,1)} & {\scriptstyle (1,1)_{\gamma=1}} \\ {\scriptstyle (A)} & {\scriptstyle (B)} & {\scriptstyle (C)} \end{bmatrix} \left\langle \begin{smallmatrix} (\lambda,\mu) \\ (D') \end{smallmatrix} \middle| J^{(1,1)}_{(A)} J^{(1,1)}_{(B)} \middle| \begin{smallmatrix} (\lambda,\mu) \\ (D) \end{smallmatrix} \right\rangle = \left\langle \begin{smallmatrix} (\lambda,\mu) \\ (D') \end{smallmatrix} \middle| J'^{(1,1)}_{(C)} \middle| \begin{smallmatrix} (\lambda,\mu) \\ (D) \end{smallmatrix} \right\rangle \\
&\qquad = \left\langle (\lambda,\mu) \middle\| J'^{(1,1)} \middle\| (\lambda,\mu) \right\rangle \begin{bmatrix} {\scriptstyle (\lambda,\mu)} & {\scriptstyle (1,1)} & {\scriptstyle (\lambda,\mu)_{\gamma=1}} \\ {\scriptstyle (D)} & {\scriptstyle (C)} & {\scriptstyle (D')} \end{bmatrix};
\end{aligned}
\tag{B.13b}
$$

$$
\begin{aligned}
&\begin{bmatrix} {\scriptstyle (1,1)} & {\scriptstyle (1,1)} & {\scriptstyle (1,1)_{\gamma=2}} \\ {\scriptstyle (A)} & {\scriptstyle (B)} & {\scriptstyle (C)} \end{bmatrix} \left\langle \begin{smallmatrix} (\lambda,\mu) \\ (D') \end{smallmatrix} \middle| J^{(1,1)}_{(A)} J^{(1,1)}_{(B)} \middle| \begin{smallmatrix} (\lambda,\mu) \\ (D) \end{smallmatrix} \right\rangle = \left\langle \begin{smallmatrix} (\lambda,\mu) \\ (D') \end{smallmatrix} \middle| J''^{(1,1)}_{(C)} \middle| \begin{smallmatrix} (\lambda,\mu) \\ (D) \end{smallmatrix} \right\rangle \\
&\qquad = \left\langle (\lambda,\mu) \middle\| J''^{(1,1)} \middle\| (\lambda,\mu)_{\gamma=1} \right\rangle \begin{bmatrix} {\scriptstyle (\lambda,\mu)} & {\scriptstyle (1,1)} & {\scriptstyle (\lambda,\mu)_{\gamma=1}} \\ {\scriptstyle (D)} & {\scriptstyle (C)} & {\scriptstyle (D')} \end{bmatrix} \\
&\qquad + \left\langle (\lambda,\mu) \middle\| J''^{(1,1)} \middle\| (\lambda,\mu)_{\gamma=2} \right\rangle \begin{bmatrix} {\scriptstyle (\lambda,\mu)} & {\scriptstyle (1,1)} & {\scriptstyle (\lambda,\mu)_{\gamma=2}} \\ {\scriptstyle (D)} & {\scriptstyle (C)} & {\scriptstyle (D')} \end{bmatrix}.
\end{aligned}
\tag{B.13c}
$$

Here we choose $\left\langle (\lambda,\mu) \middle\| J'^{(1,1)} \middle\| (\lambda,\mu)_{\gamma=2} \right\rangle = 0$.

The exact expressions of the submatrix elements follow from the equations (B.13):

$$\left\langle (\lambda,\mu) \left\| \, J^{(0,0)} \right\| (\lambda,\mu) \right\rangle = -\frac{1}{2\sqrt{2}}\, C_2^{\mathrm{SU(3)}}(\lambda,\mu); \tag{B.14a}$$

$$\left\langle (\lambda,\mu) \left\| \, J^{(1,1)} \right\| (\lambda,\mu) \right\rangle = \sqrt{C_2^{\mathrm{SU(3)}}(\lambda,\mu)}; \tag{B.14b}$$

$$\left\langle (\lambda,\mu) \left\| \, J'^{(1,1)} \right\| (\lambda,\mu) \right\rangle = -\frac{\sqrt{3}}{2}\sqrt{C_2^{\mathrm{SU(3)}}(\lambda,\mu)}; \tag{B.14c}$$

$$\left\langle (\lambda,\mu) \left\| \, J''^{(1,1)} \right\| (\lambda,\mu)_{\gamma=1} \right\rangle = -\frac{\sqrt{3}}{2\sqrt{5}}\frac{C_3^{\mathrm{SU(3)}}(\lambda,\mu)}{\sqrt{C_2^{\mathrm{SU(3)}}(\lambda,\mu)}}; \tag{B.14d}$$

$$\left\langle (\lambda,\mu) \left\| \, J''^{(1,1)} \right\| (\lambda,\mu)_{\gamma=2} \right\rangle = \frac{1}{2\sqrt{5}}\left(\frac{\lambda\mu(2+\lambda)(2+\mu)(1+\lambda+\mu)(3+\lambda+\mu)}{C_2^{\mathrm{SU(3)}}(\lambda,\mu)}\right)^{\frac{1}{2}}; \tag{B.14e}$$

where $C_2^{\mathrm{SU(3)}}(\lambda,\mu)$ and $C_3^{\mathrm{SU(3)}}(\lambda,\mu)$ are the eigenvalues of the quadratic and cubic Casimir operators of SU(3):

$$C_2^{\mathrm{SU(3)}}(\lambda,\mu) = \frac{1}{3}\left(\lambda^2+\mu^2+\lambda\mu+3\lambda+3\mu\right); \tag{B.15}$$

$$C_3^{\mathrm{SU(3)}}(\lambda,\mu) = \frac{1}{9}(\lambda-\mu)(2\lambda+\mu+3)(2\mu+\lambda+3). \tag{B.16}$$

The trace of two group generators is expressed by the formula

$$\mathrm{Tr}\left\langle \begin{smallmatrix}(\lambda,\mu)\\(D)\end{smallmatrix} \,\middle|\, J^{(1,1)}_{(A)} J^{(1,1)}_{(B)} \,\middle|\, \begin{smallmatrix}(\lambda,\mu)\\(D)\end{smallmatrix} \right\rangle = (-1)^A \frac{1}{8}\dim(\lambda,\mu) C_2^{\mathrm{SU(3)}}(\lambda,\mu)\delta_{(A)(-B)}, \tag{B.17}$$

where $\dim(\lambda,\mu) = \frac{1}{2}(\lambda+1)\mu+1)(\lambda+\mu+2)$ is the dimension of the irrep.

The trace of three group generators is calculated by using the series of CG coefficients (B.12) and the expressions of the submatrix elements (B.14)

$$\begin{aligned}\mathrm{Tr}\left\langle \begin{smallmatrix}(\lambda,\mu)\\(D)\end{smallmatrix} \,\middle|\, J^{(1,1)}_{(A)} J^{(1,1)}_{(B)} J^{(1,1)}_{(C)} \,\middle|\, \begin{smallmatrix}(\lambda,\mu)\\(D)\end{smallmatrix} \right\rangle &= -(-1)^{Z_C+M_C}\frac{\sqrt{3}}{16}\dim(\lambda,\mu) \\ \times \left\{ \begin{bmatrix} (1,1) & (1,1) & (1,1)_{\gamma=1} \\ (A) & (B) & (-C) \end{bmatrix} C_2^{\mathrm{SU(3)}}(\lambda,\mu) \right. &+ \left. \begin{bmatrix} (1,1) & (1,1) & (1,1)_{\gamma=2} \\ (A) & (B) & (-C) \end{bmatrix} \frac{1}{\sqrt{5}}\, C_3^{\mathrm{SU(3)}}(\lambda,\mu) \right\}. \end{aligned} \tag{B.18}$$

The properties of the wave functions of the SU(3) group and the transformations of the matrices are similar to the SU(2) case

$$\left| \begin{matrix}(\lambda,\mu)\\ Z,I,M\end{matrix} \right\rangle^{\dagger} = (-1)^{Z+M}\left| \begin{matrix}(\mu,\lambda)\\ -Z,I,-M\end{matrix} \right\rangle, \tag{B.19a}$$

$$D^{(\lambda,\mu)}_{(Z,I,M)(Z',I',M')}(\alpha) = (-1)^{Z+M-Z'-M'} \overset{+}{D}{}^{(\mu,\lambda)}_{(-Z',I',-M')(-Z,I,-M)}(\alpha). \tag{B.19b}$$

According to the Wigner–Eckart theorem the differential forms of the SU(3) Wigner matrices are expressed as follows:

$$\begin{aligned}
\partial_i D^{(\lambda,\mu)}_{(z,j,m)(z',j',m')}(q) &= \frac{\partial}{\partial q^i} D^{(\lambda,\mu)}_{(z,j,m)(z',j',m')}(q) \\
&= C_i^{(Z,I,M)}(q) \left\langle \begin{matrix} (\lambda,\mu) \\ z,j,m \end{matrix} \middle| J^{(1,1)}_{(Z,I,M)} \middle| \begin{matrix} (\lambda,\mu) \\ z'',j'',m'' \end{matrix} \right\rangle D^{(\lambda,\mu)}_{(z'',j'',m'')(z',j',m')}(q) \\
&= C_i'^{(Z,I,M)}(q) D^{(\lambda,\mu)}_{(z,j,m)(z'',j'',m'')}(q) \left\langle \begin{matrix} (\lambda,\mu) \\ z'',j'',m'' \end{matrix} \middle| J^{(1,1)}_{(Z,I,M)} \middle| \begin{matrix} (\lambda,\mu) \\ z',j',m' \end{matrix} \right\rangle;
\end{aligned} \tag{B.20a}$$

$$\begin{aligned}
\partial_i D^{(\lambda,\mu)}_{(z,j,m)(z',j',m')}(-q) &= \frac{\partial}{\partial q^i} D^{(\lambda,\mu)}_{(z,j,m)(z',j',m')}(-q) \\
&= -\, C_i^{(Z,I,M)}(q) D^{(\lambda,\mu)}_{(z,j,m)(z'',j'',m'')}(-q) \left\langle \begin{matrix} (\lambda,\mu) \\ z'',j'',m'' \end{matrix} \middle| J^{(1,1)}_{(Z,I,M)} \middle| \begin{matrix} (\lambda,\mu) \\ z',j',m' \end{matrix} \right\rangle \\
&= -\, C_i'^{(Z,I,M)}(q) \left\langle \begin{matrix} (\lambda,\mu) \\ z,j,m \end{matrix} \middle| J^{(1,1)}_{(Z,I,M)} \middle| \begin{matrix} (\lambda,\mu) \\ z'',j'',m'' \end{matrix} \right\rangle D^{(\lambda,\mu)}_{(z'',j'',m'')(z',j',m')}(-q);
\end{aligned} \tag{B.20b}$$

$$\begin{aligned}
&\left(\partial_i D^{(\lambda,\mu)}_{(z,j,m)(z'',j'',m'')}(q)\right) D^{(\lambda,\mu)}_{(z'',j'',m'')(z',j',m')}(-q) = \\
&\qquad = C_i^{(Z,I,M)}(q) \left\langle \begin{matrix} (\lambda,\mu) \\ z,j,m \end{matrix} \middle| J^{(1,1)}_{(Z,I,M)} \middle| \begin{matrix} (\lambda,\mu) \\ z',j',m' \end{matrix} \right\rangle;
\end{aligned} \tag{B.20c}$$

$$\begin{aligned}
&D^{(\lambda,\mu)}_{(z,j,m)(z'',j'',m'')}(-q) \left(\partial_i D^{(\lambda,\mu)}_{(z'',j'',m'')(z',j',m')}(q)\right) = \\
&\qquad = C_i'^{(Z,I,M)}(q) \left\langle \begin{matrix} (\lambda,\mu) \\ z,j,m \end{matrix} \middle| J^{(1,1)}_{(Z,I,M)} \middle| \begin{matrix} (\lambda,\mu) \\ z',j',m' \end{matrix} \right\rangle.
\end{aligned} \tag{B.20d}$$

For the derivation of the canonical momenta it is sufficient to restrict the consideration to the terms of second order in the velocities:

$$\begin{aligned}
&\int \sin\theta \mathrm{d}\theta \mathrm{d}\varphi \,\mathrm{Tr}\Big(\dot{U}\overset{+}{\dot{U}}\dot{U}\overset{+}{\dot{U}}\Big) \approx \\
&= \frac{\pi}{4} \dim(\lambda,\mu) C_2^{\mathrm{SU(3)}}(\lambda,\mu) \bigg(\frac{4}{3} \sum_M (-1)^M \left\{\dot{q}^i, C_i'^{(0,1,M)}(q)\right\} \left\{\dot{q}^{i'}, C_{i'}'^{(0,1,-M)}(q)\right\} \sin^2 F \\
&\quad + \sum_{Z,M} (-1)^{Z+M} \left\{\dot{q}^i, C_i'^{(Z,\frac{1}{2},M)}(q)\right\} \left\{\dot{q}^{i'}, C_{i'}'^{(-Z,\frac{1}{2},-M)}(q)\right\} (1-\cos F) \bigg);
\end{aligned} \tag{B.21}$$

$$\begin{aligned}
&\int \sin\theta \mathrm{d}\theta \mathrm{d}\varphi \,\mathrm{Tr}\left(A\Big[\overset{+}{A}\dot{A}, \partial_k U_0 \overset{+}{U_0}\Big]\Big[\overset{+}{A}\dot{A}, \partial^k U_0 \overset{+}{U_0}\Big] \overset{+}{A} \right) \approx \\
&\approx \int \sin\theta \mathrm{d}\theta \mathrm{d}\varphi \,\mathrm{Tr}\left(A U_0 \Big[\overset{+}{A}\dot{A}, \overset{+}{U_0}\partial_k U_0\Big]\Big[\overset{+}{A}\dot{A}, \overset{+}{U_0}\partial^k U_0\Big] \overset{+}{U_0}\overset{+}{A} \right) \\
&\approx \frac{3}{2^6} \dim(\lambda,\mu) C_2^{\mathrm{SU(3)}}(\lambda,\mu)(-1)^{Z-M'+1} \left\{\dot{q}^l, C_l'^{(Z,I,M)}(q)\right\} \left\{\dot{q}^{l'}, C_{l'}'^{(-Z,I,M')}(q)\right\} \\
&\quad \times \frac{2}{3} \int \sin\theta \mathrm{d}\theta \mathrm{d}\varphi D^I_{-M',m'}(-\alpha) \left(\partial_k D^I_{-m',m''}(\alpha)\right) D^I_{-m'',m}(-\alpha) \left(\partial^k D^I_{m,M}(\alpha)\right) =
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{\pi}{8}\dim(\lambda,\mu)C_2^{\mathrm{SU(3)}}(\lambda,\mu) \\
&\times \left(\frac{8}{3}\sum_M (-1)^M \left\{\dot{q}^i, C_i'^{(0,1,M)}(q)\right\}\left\{\dot{q}^{i'}, C_{i'}'^{(0,1,-M)}(q)\right\}\left(F'^2 + \frac{2}{r^2}\sin^2 F\right)\right. \\
&\left. + \sum_{Z,M}(-1)^{Z+M}\left\{\dot{q}^i, C_i'^{(Z,\frac{1}{2},M)}(q)\right\}\left\{\dot{q}^{i'}, C_{i'}'^{(-Z,\frac{1}{2},-M)}(q)\right\}\left(F'^2 + \frac{2}{r^2}\sin^2 F\right)\right); \qquad \text{(B.22)}
\end{aligned}
$$

$$
\begin{aligned}
&\int \sin\theta \mathrm{d}\theta\mathrm{d}\varphi \,\mathrm{Tr}\left(A\left[\overset{+}{A}\dot{A}, \partial_k U_0 \overset{+}{U}_0\right]U_0\left[\overset{+}{A}\dot{A}, \overset{+}{U}_0\partial^k U_0\right]\overset{+}{U}_0\overset{+}{A}\right) \approx \\
&\approx \int \sin\theta \mathrm{d}\theta\mathrm{d}\varphi \,\mathrm{Tr}\left(AU_0\left[\overset{+}{A}\dot{A}, \overset{+}{U}_0\partial_k U_0\right]\overset{+}{U}_0\left[\overset{+}{A}\dot{A}, \partial^k U_0\overset{+}{U}_0\right]\overset{+}{A}\right) \\
&\approx -\frac{\pi}{8}\dim(\lambda,\mu)C_2^{\mathrm{SU(3)}}(\lambda,\mu) \\
&\times\left(\frac{8}{3}\sum_M(-1)^M\left\{\dot{q}^i, C_i'^{(0,1,M)}(q)\right\}\left\{\dot{q}^{i'}, C_{i'}'^{(0,1,-M)}(q)\right\}\left(\cos 2F\cdot F'^2 + \frac{2}{r^2}\sin^2 F\cos^2 F\right)\right. \\
&\left.+\sum_{Z,M}(-1)^{Z+M}\left\{\dot{q}^i, C_i'^{(Z,\frac{1}{2},M)}(q)\right\}\left\{\dot{q}^{i'}, C_{i'}'^{(-Z,\frac{1}{2},-M)}(q)\right\}\left(\cos F\cdot F'^2 + \frac{1}{r^2}\sin 2F\sin F\right)\right).
\end{aligned}
$$
(B.23)

For the derivation of the Lagrangian density the following expressions are needed:

$$
\sum_M(-1)^{\frac{1}{2}-M}\left\{J^{(1,1)}_{(\frac{1}{2},\frac{1}{2},M)}, J^{(1,1)}_{(-\frac{1}{2},\frac{1}{2},-M)}\right\} = -\hat{C}_2^{\mathrm{SU(3)}} + \hat{C}^{\mathrm{SU(2)}} + \left(J^{(1,1)}_{(0,0,0)}\right)^2, \qquad \text{(B.24)}
$$

$$
\sum_{M,M'}\begin{bmatrix}1 & 1 & 1\\ M & M' & u\end{bmatrix}J^{(1,1)}_{(0,1,M)}J^{(1,1)}_{(0,1,M')} = -\frac{1}{\sqrt{2}}\,J^{(1,1)}_{(0,1,u)}, \qquad \text{(B.25)}
$$

$$
\left[\left[J^{(1,1)}_{(0,1,\cdot)}\times\hat{x}\right]_u, \left(J^{(1,1)}_{(0,1,\cdot)}\cdot\hat{x}\right)\right] = 2\mathrm{i}\left(J^{(1,1)}_{(0,1,u)} - \hat{x}_u\left(J^{(1,1)}_{(0,1,\cdot)}\cdot\hat{x}\right)\right), \qquad \text{(B.26)}
$$

$$
\begin{aligned}
&E^{(\bar{A})(\bar{B})}(F)\,J^{(1,1)}_{(\bar{A})}J^{(1,1)}_{(\bar{B})} = \\
&= -\frac{1}{a_{\frac{1}{2}}(F)}\hat{C}_2^{\mathrm{SU(3)}} + \left(\frac{1}{a_{\frac{1}{2}}(F)} - \frac{1}{a_1(F)}\right)\hat{C}^{\mathrm{SU(2)}} + \frac{1}{a_{\frac{1}{2}}(F)}\left(J^{(1,1)}_{(0,0,0)}\right)^2, \qquad \text{(B.27)}
\end{aligned}
$$

$$
\begin{aligned}
&E^{(\bar{A})(\bar{B})}(F)D^I_{(\bar{B}')(\bar{B})}(\hat{x},F(r))\,J^{(1,1)}_{(\bar{A})}J^{(1,1)}_{(\bar{B}')} = \\
&= -\frac{\cos F}{a_{\frac{1}{2}}(F)}\hat{C}_2^{\mathrm{SU(3)}} + \left(\frac{\cos F}{a_{\frac{1}{2}}(F)} - \frac{\cos 2F}{a_1(F)}\right)\hat{C}^{\mathrm{SU(2)}} + \frac{\cos F}{a_{\frac{1}{2}}(F)}\left(J^{(1,1)}_{(0,0,0)}\right)^2 \\
&+\mathrm{i}\left(\frac{\sin 2F}{a_1(F)} + \frac{\sin F}{a_{\frac{1}{2}}(F)}\right)\left(J^{(1,1)}_{(0,1,\cdot)}\cdot\hat{x}\right) - 2\frac{\sin^2 F}{a_1(F)}\left(J^{(1,1)}_{(0,1,\cdot)}\cdot\hat{x}\right)\left(J^{(1,1)}_{(0,1,\cdot)}\cdot\hat{x}\right). \qquad \text{(B.28)}
\end{aligned}
$$

Here $D^{I}_{(\bar{B}')(\bar{B})}(\hat{x}, F(r))$ is a Wigner matrix of the SU(2). The summation is over the SU(2) representations $I = \frac{1}{2}$ and 1, and the corresponding bases

$$E^{(\bar{A})(\bar{B})}(F) \begin{bmatrix} {}^{(1,1)}_{(\bar{A})} & {}^{(1,1)}_{(0,1,u)} & {}^{(1,1)_a}_{(\bar{C})} \end{bmatrix} J^{(1,1)}_{(\bar{B})} J^{(1,1)}_{(\bar{C})} = \frac{1}{\sqrt{3}} \left(\frac{1}{2a_{\frac{1}{2}}(F)} + \frac{1}{a_1(F)} \right) J^{(1,1)}_{(0,1,u)}, \tag{B.29}$$

$$\begin{aligned} &E^{(\bar{A})(\bar{B})}(F) \begin{bmatrix} {}^{(1,1)}_{(\bar{A})} & {}^{(1,1)}_{(0,1,u)} & {}^{(1,1)_a}_{(\bar{C})} \end{bmatrix} D^{I}_{(\bar{C}')(\bar{C})}(\hat{x}, F(r)) J^{(1,1)}_{(\bar{B})} J^{(1,1)}_{(\bar{C}')} = \\ &= \frac{1}{\sqrt{3}} \Bigg\{ - \left[J^{(1,1)}_{(0,1,\cdot)} \times \hat{x} \right]_u \left(\frac{\sin F}{2a_{\frac{1}{2}}(F)} + \mathrm{i} \frac{2\sin^2 F}{a_1(F)} \left(J^{(1,1)}_{(0,1,\cdot)} \cdot \hat{x} \right) \right) \\ &\qquad - \mathrm{i} \left(\frac{\sin F}{2a_{\frac{1}{2}}(F)} \hat{C}_2^{\mathrm{SU(3)}} - \left(\frac{\sin F}{2a_{\frac{1}{2}}(F)} - \frac{\sin 2F}{a_1(F)} \right) \hat{C}^{\mathrm{SU(2)}} - \frac{\sin F}{2a_{\frac{1}{2}}(F)} \left(J^{(1,1)}_{(0,0,0)} \right)^2 \right) \hat{x}_u \\ &\qquad + \left(\frac{\cos 2F}{a_1(F)} + \frac{\cos F}{2a_{\frac{1}{2}}(F)} + \mathrm{i} \frac{\sin 2F}{a_1(F)} \left(J^{(1,1)}_{(0,1,\cdot)} \cdot \hat{x} \right) \right) J^{(1,1)}_{(0,1,u)} \Bigg\}. \end{aligned} \tag{B.30}$$

Appendix C. Definitions for the noncanonical SU(3) soliton

The antisymmetrical isoscalar factors are:

$$\begin{bmatrix} (1,1) & (1,1) & (1,1)_{\gamma=1} \\ 1 & 1 & 1 \end{bmatrix} = \frac{1}{\sqrt{2\cdot 3}}; \qquad \begin{bmatrix} (1,1) & (1,1) & (1,1)_{\gamma=1} \\ 2 & 2 & 1 \end{bmatrix} = -\frac{\sqrt{5}}{\sqrt{2\cdot 3}};$$
$$\begin{bmatrix} (1,1) & (1,1) & (1,1)_{\gamma=1} \\ 2 & 1 & 2 \end{bmatrix} = \frac{1}{\sqrt{2}}; \qquad \begin{bmatrix} (1,1) & (1,1) & (1,1)_{\gamma=1} \\ 1 & 2 & 2 \end{bmatrix} = \frac{1}{\sqrt{2}}. \tag{C.1}$$

The symmetrical isoscalar are:

$$\begin{bmatrix} (1,1) & (1,1) & (1,1)_{\gamma=2} \\ 2 & 2 & 2 \end{bmatrix} = -\frac{\sqrt{7}}{\sqrt{2\cdot 5}}; \qquad \begin{bmatrix} (1,1) & (1,1) & (1,1)_{\gamma=2} \\ 1 & 1 & 2 \end{bmatrix} = -\frac{\sqrt{3}}{\sqrt{2\cdot 5}};$$
$$\begin{bmatrix} (1,1) & (1,1) & (1,1)_{\gamma=2} \\ 1 & 2 & 1 \end{bmatrix} = \frac{1}{\sqrt{2}}; \qquad \begin{bmatrix} (1,1) & (1,1) & (1,1)_{\gamma=2} \\ 2 & 1 & 1 \end{bmatrix} = \frac{1}{\sqrt{2}}. \tag{C.2}$$

The submatrix elements are expressed by using a series of the Clebsch-Gordan coefficients (B.12), some of which have exact values:

$$\begin{aligned} \left\langle (1,0) \left\| J^{(0,0)}_{\mathrm{SO(3)}} \right\| (1,0) \right\rangle &= \frac{8}{3\sqrt{2}}; \\ \left\langle (1,0) \left\| J^{(1,1)}_{\mathrm{SO(3)}} \right\| (1,0) \right\rangle &= \frac{4}{\sqrt{3}}; \\ \left\langle (1,0) \left\| J'^{(1,1)}_{\mathrm{SO(3)}} \right\| (1,0) \right\rangle &= -\,4; \\ \left\langle (1,0) \left\| J''^{(1,1)}_{\mathrm{SO(3)}} \right\| (1,0) \right\rangle &= -\,\frac{4\sqrt{5}}{3}. \end{aligned} \tag{C.3}$$

The trace of two group generators is expressed by the formula

$$\mathrm{Tr}\left\langle {(1,0) \atop (D)} \middle| J_{(L_1,M_1)} J_{(L_2,M_2)} \middle| {(1,0) \atop (D)} \right\rangle = (-1)^{M_1} 2\, \delta_{L_1,L_2} \delta_{M_1,-M_2}\,. \tag{C.4}$$

The trace of three group generators is expressed by the formula

$$\begin{aligned} \mathrm{Tr}\left\langle {(1,0) \atop (D)} \middle| J_{(A)} J_{(B)} J_{(C)} \middle| {(1,0) \atop (D)} \right\rangle = (-1)^{M_C+1} 2\sqrt{3} \Bigg\{ & \begin{bmatrix} (1,1) & (1,1) & (1,1)_{\gamma=1} \\ (A) & (B) & (-C) \end{bmatrix} \\ & + \frac{\sqrt{5}}{3} \begin{bmatrix} (1,1) & (1,1) & (1,1)_{\gamma=2} \\ (A) & (B) & (-C) \end{bmatrix} \Bigg\}. \end{aligned} \tag{C.5}$$

The trace of four group generators is expressed by the formula

$$\begin{aligned}\mathrm{Tr}\left\langle {}^{(1,0)}_{(D)} \middle| J_{(L_1,M_1)} J_{(L_2,M_2)} J_{(L_3,M_3)} J_{(L_4,M_4)} \middle| {}^{(1,0)}_{(D)} \right\rangle &= 4\Big[(2L_1+1)(2L_2+1) \\ &\times (2L_3+1)(2L_4+1)\Big]^{\frac{1}{2}} \sum_k (-1)^u \begin{Bmatrix} L_1 & L_2 & k \\ 1 & 1 & 1 \end{Bmatrix} \begin{Bmatrix} L_3 & L_4 & k \\ 1 & 1 & 1 \end{Bmatrix} \\ &\times \begin{bmatrix} L_1 & L_2 & k \\ M_1 & M_2 & -u \end{bmatrix} \begin{bmatrix} L_3 & L_4 & k \\ M_3 & M_4 & u \end{bmatrix}. \end{aligned} \tag{C.6}$$

Appendix D. Calculations with computer algebra system

The basis states in a general SU(3) irrep (λ, μ) are specified by the parameters (Z, I, M), where the hypercharge is $Y = \frac{2}{3}(\mu - \lambda) - 2Z$. However expressions with sums over the parameters (Z, I, M) are inconvenient for consideration. All basis vectors of SU(3) are distributed in the grating which is restricted by ABCDEF hexagon, and the sides of which are described only by the parameters λ and μ (see Fig. 4).

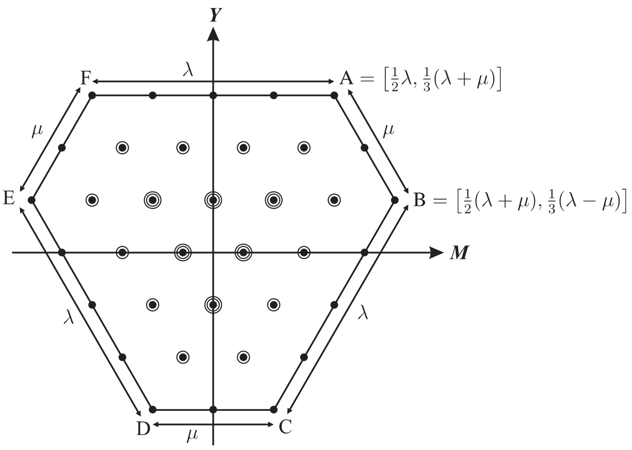

Figure 4: The space of the basis vectors of the SU(3) representation. The concentric circles indicate the number of independent basis vectors with the same Y and M but different isospin I [112, 113].

To fulfil a symbolic summation over all basis vectors inside the hexagon for our expressions we use the computer algebra system MATHEMATICA. The algorithm of summation was created considering inequalities (II.1.5), which are satisfied by the basis state parameters. Summation from the minimal to maximal isospin and hypercharge values is performed by the following algorithm:

$$\mathbf{(Expression)/.}\left\{\mathbf{I} \to \frac{\mathbf{x+y}}{\mathbf{2}}, \mathbf{Z} \to \frac{\mathbf{y-x}}{\mathbf{2}}\right\};$$
$$\mathbf{Sum}\left[\%, \{\mathbf{x}, 0, \mu\}, \{\mathbf{y}, 0, \lambda\}, \left\{\mathbf{M}, \frac{\mathbf{x+y}}{\mathbf{2}}, -\frac{\mathbf{x+y}}{\mathbf{2}}\right\}\right] \ \mathbf{//Fullsimplify.}$$

These above simple commands lead to the correct sum in the expressions without denominators. Using partial summation it is possible to calculate the expressions with denominators:

$$\mathbf{Function[\,x\_, Z\_, M\_\,] := \Big((Expression)/.}\left\{\mathbf{I} \to \frac{\mathbf{x}}{\mathbf{2}}\right\}\Big);$$
$$\Big(\mathbf{Sum}\Big[\mathbf{Function[\,x, Z, M\,]}, \{\mathbf{x}, \mu, \lambda+\mu\}, \left\{\mathbf{Z}, \frac{\mathbf{x}}{\mathbf{2}} - \lambda, \mu - \frac{\mathbf{x}}{\mathbf{2}}\right\}, \left\{\mathbf{M}, -\frac{\mathbf{x}}{\mathbf{2}}, \frac{\mathbf{x}}{\mathbf{2}}\right\}\Big]$$

```
Sum[Function[x,Z,M],{x,λ+1,μ-1},{Z,x/2-λ,x/2},{M,-x/2,x/2}]
Sum[Function[x,Z,M],{x,0,λ},{Z,-x/2,x/2},{M,-x/2,x/2}])//Fullsimplify.
```

By using the actions (II.1.6) of the generators on the basis states, we use the MATHEMATICA to find the matrix elements of the SU(3) group generators in any representation. The expressions of group generators traces are derived from the actions (II.1.6) by using the appropriate permutations.

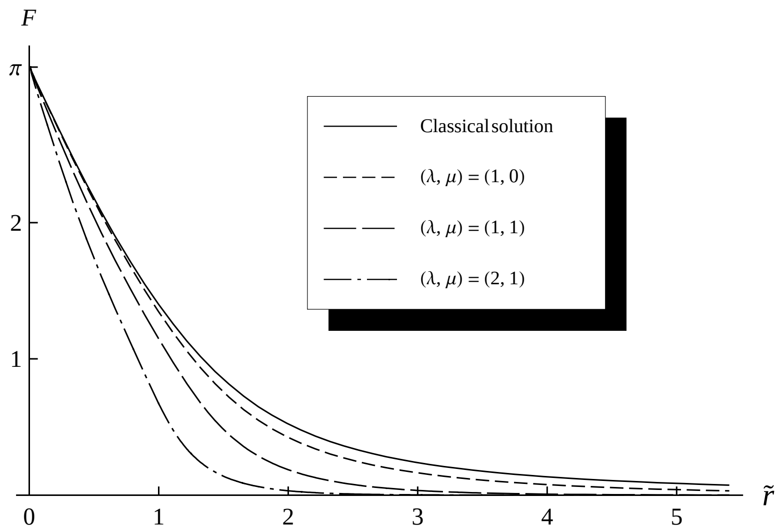

Figure 5: Solutions of the profile function $F(\tilde{r})$ for the $B = 1$ skyrmion in various SU(3) representations.

Numerical calculations of the integro-differential equation (II.6.3) can be performed in the following way.

1. Using the classical profile function (see Fig. 2) and any pair of empirical baryon observables, for example, the nucleon mass (II.2.8) and the isoscalar radius $\langle r^2 \rangle = -\frac{2}{\pi e^2 f_\pi^2} \int r^2 F' \sin^2 F \mathrm{d}r$, we fit two model parameters f_π and e and calculate all required integrals in the quantum equation (II.6.3).

2. Using the known asymptotic solution (II.6.7) (and its derivative) we can adopt a simple procedure solving the differential equation and find the first approximation of the quantum solution $F_1(\tilde{r})$ and the constant k_1 in (II.6.7).

3. The obtained function $F_1(\tilde{r})$ can be used to recalculate f_π, e and the integrals. The procedure described in item 2 can be used again to get the second approximation to the quantum solution $F_2(\tilde{r})$ and the constant k_2.

4. This procedure can be iterated until the convergent solution and the parameters f_π, e as well as stable values of $M_{\rm cl}(F), \Delta M_k(F), a_k(F), \tilde{m}(F)$ are obtained. The self-consistent set then can be used to calculate numerous phenomenologically interesting quantities.

Quantum profile function solutions with the model parameters determined from the nucleon mass and the isoscalar radius are shown in Fig. 5. A successful in initial guess requires only 10–20 iterations to get 5–6 fixed digits in all integrals and f_π, e values.

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Santrauka

Disertaciją sudaro disertanto mokslinių darbų sąrašas, įvadas, nagrinėjamo modelio teorinė apžvalga, trys pagrindiniai skyriai, išvados, keturi priedai, kuriuose pateikiamos skaičiavimuose naudotos pagalbinės išraiškos ir cituojamos literatūros sąrašas.

Įvade trumpai supažindinama su nagrinėjamu modeliu, pagrindžiamas darbo aktualumas, suformuluojami darbo tikslai ir mokslinis naujumas, pristatomi disertacijos ginamieji teiginiai ir pateikiama darbo rezultatų aprobacija.

Pirmo skyriaus pradžioje pateikiama trumpa Skyrme'os modelio istorinė apžvalga. Toliau aprašomas netiesinis sigma modelis, kurio pagrindu buvo sukonstruotas Skyrme'os modelis. Pateikiamas klasikinio Skyrme'os modelio formalizmas ir jo praplėtimas įvedant aukštesnio laipsnio lagranžiano narius. Toliau skyriuje aprašomas racionalaus atvaizdžio artinio formalizmas ir įvairūs Skyrme'os modelio apibendrinimo metodai. Šiame skyriuje taip pat aprašomas Wess-Zumino dėmens matematinis aparatas ir Skyrme'os modelio kvantavimas.

Antrame skyriuje pateikiamas Skyrme'os modelio apibendrinimas bet kuriam SU(3) grupės neredukuotiniam įvaizdžiui (λ, μ). Skyriaus pradžioje apibrėžiamas modelio unitarusis laukas, sukonstruojami SU(3) grupės generatoriai ir užrašomas jų veikimas į bazines būsenas. Toliau pateikiamas SU(3) klasikinio Skyrme'os modelio formalizmas ir kanoninio kvantavimo procedūra. Kvantuojant įvedamos kvantinės kolektyvinės koordinatės, išvedamos metrinio tenzoriaus ir solitono inercijos momentų išraiškos. Užrašomas kvantinis lagranžianas ir nuo įvaizdžio priklausančios kvantinės masės pataisos. Apskaičiuojamas Wess-Zumino narys ir užrašoma hamiltoniano išraiška. Skyriaus pabaigoje pateikiamas chiralinės simetrijos pažeidimo narys.

Trečiame skyriuje įvedamas Skyrme'os modelio solitoninis sprendinys, kuris apibrėžtas nekanoninėje SU(3) $\supset$ SO(3) bazėje. Skyriaus pradžioje pateikiamas klasikinis sprendinys, kanoninių ir nekanoninių bazinių vektorių sąryšiai bei nekanoninių generatorių išraiškos. Toliau nagrinėjamas modelio kvantavimas, užrašomos solitono inercijos momentų, efektyvaus lagranžiano ir kvantinių pataisų išraiškos. Skyriaus pabaigoje pateikiamos hamiltoniano tankio ir chiralinės simetrijos pažeidimo nario išraiškos.

Ketvirtame skyriuje nagrinėjamas SU(3) Skyrme'os modelio nekanoniškai įdėtas solitonas racionalaus atvaizdžio artinyje, kai barioninis krūvis $B \geq 2$. Skyriaus pradžioje pateikiamas racionalaus atvaizdžio solitoninis sprendinys, barioninio krūvio tankio išraiška ir klasikinio modelio formalizmas racionalaus atvaizdžio artinyje. Toliau, po kanoninio kvantavimo, užrašomi solitono inercijos momentai, lagranžiano tankio išraiška ir naujos kvantinės pataisos. Skyriaus pabaigoje pateikiamos hamiltoniano ir chiralinės simetrijos pažeidimo nario išraiškos.

Pagrindiniai rezultatai ir išvados:

1. Topologinis Skyrme'os modelis apibendrintas bet kuriam SU(3) grupės neredukuotiniam įvaizdžiui (λ, μ). Jeigu klasikiniam modelyje priklausomybė nuo įvaizdžio išreiškiama bendru daugikliu prieš Lagrange'o funkciją, tai kanoniškai kvantuojant kvantinės pataisos esminiai priklauso nuo įvaizdžio. Įvaizdį (λ, μ) galima traktuoti kaip naują diskretinį modelio fenomenologinį parametrą.

2. Wess-Zumino nario priklausomybė nuo neredukuotinio įvaizdžio (λ, μ) išreiškiama daugikliu, proporcingu SU(3) grupės trečiojo laipsnio Casimir'o operatoriaus tikrinei vertei. Dėl to sau sujungtiniams įvaizdžiams $(\lambda = \mu)$ Wess-Zumino narys išnyksta.

3. Simetriją pažeidžiančio nario Lagrange'o operatoriaus funkcinė priklausomybė nuo funkcijos $F(r)$ įvairuoja skirtingiems įvaizdžiams. Sau sujungtiniams įvaizdžiams simetriją pažeidžiantis narys supaprastėja iki SU(2) Skyrme'os modeliui įprastos formos.

4. Įvestas naujas Skyrme'os modelio solitoninis sprendinys, kuris apibrėžtas nekanoninėje SU(3) $\supset$ SO(3) bazėje. Kanoniškai kvantuojant gaunamos naujos kvantinių pataisų išraiškos ir du skirtingi solitono inercijos momentai, iš kurių vienas sutampa su SU(2) solitono momentu. Wess-Zumino narys nekanoniškai įdėtam SO(3) solitonui visada lygus nuliui.

5. Išnagrinėtas Skyrme'os modelio nekanoniškai įdėtas solitonas racionalaus atvaizdžio artinyje, kai barioninis krūvis $B \geq 2$. Kanoniškai kvantuojant gaunami penki skirtingi solitono inercijos momentai ir kvantinės pataisos. Aukštesniems įvaizdžiams Hamiltono operatorius nėra diagonalus nekanoninės bazės būsenų atžvilgiu. Šis artinys gali būti panaudotas aprašant lengvuosius branduolius kaip specialius solitonus.

Darius Jurčiukonis

SU(3) TOPOLOGINIŲ SOLITONŲ
KANONINIS KVANTAVIMAS

Daktaro disertacija
Fiziniai mokslai, fizika (02 P), matematinė ir bendroji teorinė fizika, klasikinė mechanika, kvantinė mechanika, reliatyvizmas, gravitacija, statistinė fizika, termodinamika (190 P)

Spausdino Vilniaus universiteto leidyklos spaustuvė
Universiteto g. 1, LT–01122 Vilnius
http://www.leidykla.vu.lt