Non-Abelian geometric phases in periodically driven systems

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(Received 14 November 2018; published 24 July 2019)

We consider a periodically driven quantum system described by a Hamiltonian which is the product of a slowly varying Hermitian operator $V(\lambda(t))$ and a dimensionless periodic function with zero average. We demonstrate that the adiabatic evolution of the system within a fully degenerate Floquet band is accompanied by non-Abelian (noncommuting) geometric phases appearing when the slowly varying parameter $\lambda = \lambda(t)$ completes a closed loop. The geometric phases can have significant values even after completing a single cycle of the slow variable. Furthermore, there are no dynamical phases masking the non-Abelian Floquet geometric phases, as the former average to zero over an oscillation period. This can be used to precisely control the evolution of quantum systems, in particular for performing qubit operations. The general formalism is illustrated by analyzing a spin in an oscillating magnetic field with arbitrary strength and a slowly changing direction.

I. INTRODUCTION

Topological and many-body properties of physical systems can be enriched by applying a periodic driving [1–11]. This extends to a wide range of condensed matter [1,3,4,12–18], photonic [19–23], and ultracold atom [24–56] systems. For example, the periodic driving can induce a nonstaggered synthetic magnetic flux [5,32,35,39,40,44,52] or facilitate the realization of the Haldane model [1,42] for ultracold atoms in optical lattices. To deal with periodically driven quantum systems, it is convenient to describe their long-term dynamics in terms of an effective time-independent Floquet Hamiltonian. In that case fast oscillations of the system within a driving period are represented by a micromotion operator. If the driving frequency exceeds other characteristic frequencies of the system, the Floquet Hamiltonian and the micromotion operator can be expanded in inverse powers of the driving frequency [52,57–65].

It is quite common that the periodic driving changes in time. For example, in typical ultracold atom experiments one ramps up the periodic driving from zero to a stationary regime [66]. In the previous paper [67] we have considered such a situation where a quantum system is subjected to a periodic driving which changes slowly in time. High-frequency expansions have been obtained for the effective Hamiltonian and for the micromotion operators showing that these operators change in time because of the changes in the periodic driving [67]. Furthermore the expanded effective Hamiltonian contains an extra second-order term appearing due to the slow changes in the periodic driving [67]. This can provide non-Abelian (noncommuting) geometric phases for periodically driven systems.

The high-frequency expansion was applied to a spin in a fast oscillating magnetic field with a slowly changing amplitude [67]. If the magnetic field slowly changes its direction performing a cyclic evolution in three-dimensional space, a non-Abelian geometric phase appears after the slow variable (the magnetic field amplitude) completes a cycle. Yet the acquired phase represents a small second-order correction, so the slow variable should complete many cycles to accumulate a substantial geometric phase. This is because the high-frequency expansion is applicable only if the driving strength is small compared to the driving frequency. The current analysis does not rely on such an approximation for the periodic driving. We show that the system can acquire substantial geometric phases even if the slow variable completes just a single cycle.

We study a periodically driven quantum system characterized by a Hamiltonian which is a product of a slowly varying Hermitian operator and a fast oscillating periodic function with a zero average. We transform the equations of motion to a new representation by applying a time-dependent unitary transformation. The transformation eliminates the original Hamiltonian in the equations of motion, and there is an extra term due to the slow changes in the periodic driving. Neglecting the latter term, individual Floquet bands are completely degenerate. The slow changes of the driving couples the Floquet states. We apply the adiabatic approximation by neglecting the coupling between different Floquet bands separated by the driving frequency times an integer. This is equivalent to the zero order of the high-frequency expansion [67] of the Floquet effective Hamiltonian in the transformed representation. It is demonstrated that the adiabatic evolution of the system within an individual degenerate Floquet band is accompanied by non-Abelian geometric phases which can be sufficiently large even after completing a single cycle of the slow variable. Furthermore, there are no dynamical phases masking the non-Abelian Floquet geometric phases, as the former average to zero over an oscillation period. This can be used for precisely controlling the evolution of quantum systems, in particular for performing qubit operations.

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The paper is organized as follows. In Sec. II we define a periodically driven system with a slowly modulated driving and go to a new representation via a time-dependent unitary transformation. In Sec. III we consider the adiabatic evolution of the system within an individual Floquet band and show that the evolution is accompanied by non-Abelian geometric phases. In Sec. IV we analyze the operator responsible for the geometric phases and provide explicit expressions for this operator in specific situations. Section V illustrates the general formalism by analyzing a spin in an oscillating magnetic field with arbitrary strength and a slowly changing direction. The concluding Sec. VI summarizes the findings. Technical details of some calculations are presented in the two appendices.

II. PERIODICALLY DRIVEN SYSTEM WITH A MODULATED DRIVING

A. Hamiltonian and equations of motion

We consider a periodically driven (Floquet) quantum system with a slowly modulated driving. The system is described by a Hamiltonian which is the product of a slowly varying Hermitian operator $V(\lambda) = V(\lambda(t))$ and a fast oscillating dimensionless real function $f(\omega t + \theta)$:

$$H(\omega t + \theta, t) = V(\lambda(t))f(\omega t + \theta), \quad (1)$$

where ω is the oscillation frequency and θ defines the phase of the oscillations. The operator $V(\lambda)$ depends on time via a set of slowly varying parameters $\lambda = \lambda(t) = \{\lambda_\mu(t)\}$ which change little over the driving period $T = 2\pi/\omega$; the subscript μ specifies individual slowly varying parameters. Here also $f(\omega t + \theta)$ is taken to be a 2π periodic function $f(\omega t + \theta + 2\pi) = f(\omega t + \theta)$ with an amplitude of the order of unity and zero average: $\int_0^{2\pi} f(\theta')d\theta' = 0$. Therefore the Fourier expansion of the Hamiltonian

$$H(\theta', t) = \sum_{m=-\infty}^{\infty} H^{(m)}(t)e^{im\theta'} \text{ with } \theta' = \omega t + \theta \quad (2)$$

does not contain a zero-frequency component, $H^{(0)}(t) = 0$, while the other components are

$$H^{(m)}(t) = V(\lambda(t))f^{(m)}, \quad (3)$$

with $f^{(m)} = \int_0^{2\pi} f(\theta')e^{-im\theta'}d\theta'$.

An example of such a system is a spin in a magnetic field $\mathbf{B}(t)f(\omega t + \theta)$ [67] with a fast oscillating amplitude $\propto f(\omega t + \theta)$ and a slowly changing direction $\propto \mathbf{B}(t)$, where $\mathbf{B}(t)$ plays the role of $\lambda(t)$. In that case the slowly varying part of the Hamiltonian is given by

$$V(\mathbf{B}(t)) = g_F \mathbf{F} \cdot \mathbf{B}(t), \quad (4)$$

where g_F is a gyromagnetic factor, $\mathbf{F} = F_1\mathbf{e}_x + F_2\mathbf{e}_y + F_3\mathbf{e}_z$ is a spin operator with Cartesian components satisfying the usual commutation relations $[F_s, F_q] = i\hbar\epsilon_{sqr}F_r$. Here ϵ_{sqr} is the Levi-Civita symbol, and a summation over a repeated Cartesian index $r = x, y, z$ is implied.

A state vector $|\phi(t)\rangle$ of the system belongs to the Hilbert space $\mathscr{H}$ and obeys the time-dependent Schrödinger equation

(TDSE):

$$i\hbar\frac{\partial}{\partial t}|\phi(t)\rangle = V(\lambda(t))f(\omega t + \theta)|\phi(t)\rangle. \quad (5)$$

Previously a general perturbative analysis was carried out to deal with the evolution of a periodically driven system with a modulated driving using the Floquet extended-space approach [67]. Such a perturbative treatment is generally valid if matrix elements of the Fourier components of the periodic Hamiltonian are small compared to the driving frequency:

$$\left|H_{\alpha\beta}^{(m)}\right| \ll \hbar\omega, \quad \text{and hence} \quad |V_{\alpha\beta}|f^{(m)} \ll \hbar\omega, \quad (6)$$

where the subscripts α and β are used to label the matrix element of the operators. Furthermore, $H_{\alpha\beta}^{(m)}$ should change sufficiently slowly over the driving period.

B. Transformed representation

In what follows we will consider the dynamics of the system when the weak driving condition (6) does not necessarily hold. For this, we will go to another representation via a unitary operator which eliminates the original Hamiltonian in the transformed equations of motion. Such a unitary operator reads

$$R(\omega t + \theta, \lambda(t)) = \exp\left[-i\frac{\mathcal{F}(\omega t + \theta)}{\hbar\omega}V(\lambda(t))\right], \quad (7)$$

where $\mathcal{F}(\theta')$ is a primitive function of $f(\theta')$ with zero average:

$$d\mathcal{F}(\theta')/d\theta' = f(\theta') \quad \text{and} \quad \int_0^{2\pi}\mathcal{F}(\theta')d\theta' = 0. \quad (8)$$

The calligraphy letter $\mathcal{F}$ is used to avoid confusion with the spin operator $\mathbf{F}$ featured in Eq. (4).

The transformed state vector

$$|\psi(t)\rangle = R^\dagger(\omega t + \theta, t)|\phi(t)\rangle \quad (9)$$

obeys the TDSE

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = W(\omega t + \theta, t)|\psi(t)\rangle, \quad (10)$$

with

$$W(\theta', t) = -i\hbar R^\dagger(\theta', \lambda(t))\partial R(\theta', \lambda(t))/\partial t, \quad (11)$$

where the partial derivative $\partial R/\partial t$ is calculated for a fixed value of the variable $\theta' = \omega t + \theta$. Alternatively, the transformed Hamiltonian can be represented as

$$W(\theta', t) = \dot{\lambda}_\mu A_\mu(\theta', \lambda), \quad (12)$$

where summation over repeated indices μ is implied, and

$$A_\mu(\theta', \lambda) = -i\hbar R^\dagger(\theta', \lambda)\frac{\partial R(\theta', \lambda)}{\partial\lambda_\mu} \quad (13)$$

is the μth component of the vector potential $\mathbf{A}(\theta', \lambda)$.

In this way, the transformation $R(\theta', \lambda(t))$ eliminates the original Hamiltonian (1) in the transformed equation of motion (10). The new Hamiltonian $W(\omega t + \theta, t)$ given by Eqs. (11) and (12) is due to the slow temporal changes of the variable $\lambda = \lambda(t)$ entering the transformation (7). Therefore the transformed Hamiltonian $W(\omega t + \theta, t)$ can be arbitrarily

small even if the original weak driving (high-frequency) condition (6) is violated. In particular, one has $W(\omega t + \theta, t) = 0$ for a pure periodic driving where $\lambda(t)$ is constant.

The evolution of the transformed state vector can be represented as

$$|\psi(t)\rangle = U(t, t_0)|\psi(t_0)\rangle, \tag{14}$$

with

$$U(t, t_0) = \mathcal{T} \exp\left[-\frac{i}{\hbar} \int_{t_0}^{t} W(\omega t' + \theta, t') dt'\right], \tag{15}$$

where $\mathcal{T}$ indicates the time ordering and t_0 is an initial time. The operator W determining the evolution of the transformed state vector will be analyzed in Sec. IV.

Like the original Hamiltonian $H(\omega t + \theta, t)$, the transformed Hamiltonian $W(\omega t + \theta, t)$ is 2π periodic with respect to the first variable and thus can be expanded in a Fourier series with respect to the fast variable:

$$W(\omega t + \theta, t) = \sum_{n=-\infty}^{\infty} W^{(n)}(t) e^{in(\omega t + \theta)}, \tag{16}$$

where

$$W^{(n)}(t) = \frac{1}{2\pi} \int_0^{2\pi} W(\theta', t) e^{-in\theta'} d\theta'. \tag{17}$$

Expanding also the transformed state vector $|\psi(t)\rangle \equiv |\psi(\omega t + \theta, t)\rangle$:

$$|\psi(\omega t + \theta, t)\rangle = \sum_{n=-\infty}^{\infty} |\psi^{(n)}(t)\rangle e^{in(\omega t + \theta)}, \tag{18}$$

the TDSE (10) provides the following equation for the slowly changing Fourier components $|\psi^{(n)}(t)\rangle$:

$$i\hbar \frac{\partial}{\partial t} |\psi^{(n)}(t)\rangle = \sum_{m=-\infty}^{\infty} K_{nm} |\psi^{(m)}(t)\rangle, \tag{19}$$

where

$$K_{nm} = n\hbar\omega\delta_{nm} + W^{(n-m)}(t) \tag{20}$$

are the matrix elements of the extended space Floquet Hamiltonian $\hat{K}(t)$ slowly changing in time [67]. Its of-diagonal terms $K_{nm} = W^{(n-m)}(t)$ with $n \neq m$ describe the coupling between different Fourier components (different Floquet bands) $|\psi^{(n)}(t)\rangle$ and $|\psi^{(m)}(t)\rangle$ due to the changes of the periodic driving. The diagonal elements $K_{nn} = n\hbar\omega + W^{(0)}(t)$ contain the energy of the nth Floquet manifold $n\hbar\omega$ and an extra operator $W^{(0)}(t)$ emerging due to the changes of the periodic driving. Figure 1 illustrates the coupling between different Floquet manifolds ($n \neq m$) and within of the same Floquet bands ($n = m$). Neglecting all coupling terms $W^{(n-m)}(t)$ in Eq. (20), the eigenstates of the operator $\hat{K}(t)$ are completely degenerate within individual Floquet bands with quasienergies $n\hbar\omega$ shown by horizontal lines in Fig. 1. When the effects due to the changes of the periodic driving are included, the emerging operator $W^{(0)}(t)$ provides the Floquet geometric phases for the adiabatic motion within a single degenerate Floquet manifold. We will consider this issue in more detail in the next section.
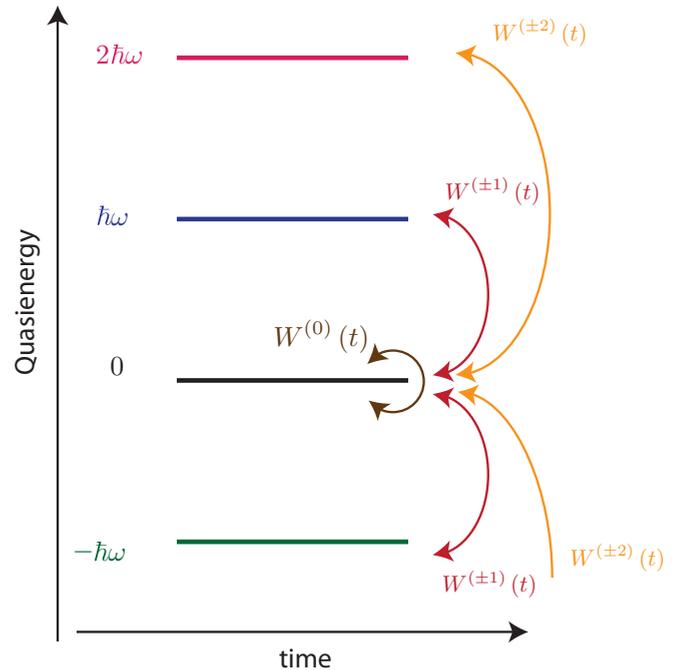

FIG. 1. Schematic representation of coupling between the Floquet bands in the transformed representation described by Eqs. (19) and (20). The operators $W^{(m)}(t)$ with $m \neq 0$ describe coupling between different Floquet manifolds, whereas the operator $W^{(0)}(t)$ couples the states belonging to the same degenerate Floquet band. The latter $W^{(0)}(t)$ provides non-Abelian geometric phases for the adiabatic motion of the system within the same Floquet manifold.

III. ADIABATIC APPROACH

A. Effective evolution operator in transformed representation

We are interested in a situation where $V(\lambda(t))$ changes sufficiently slowly, so the matrix elements of the Fourier components of $W(\omega t + \theta, t)$ are smaller than the driving frequency

$$\left|W_{\alpha\beta}^{(n)}\right| \ll \hbar\omega \tag{21}$$

and also change sufficiently smoothly. The condition (21) has the same form as the original condition (6) with H replaced by W. Since the transformed Hamiltonian $W(\omega t + \theta, t)$ given by Eqs. (11) and (12) is due to the temporal changes of $V(\lambda(t))$, the condition (21) relies on the slow changes of the periodic driving rather that on its weakness. Therefore Eq. (21) can hold even if the matrix elements $|H_{\alpha\beta}^{(n)}|$ exceed $\hbar\omega$, and thus there is a violation of the original high-frequency requirement (6).

Applying the condition (21), the evolution of the transformed state vector can be described by means of the slowly changing Floquet effective Hamiltonian $W_{\text{eff}}(t)$ expanded in the inverse powers of the driving frequency $\omega^{(-n)}$ (with $n \geqslant 0$) [67]. The adiabatic approximation is obtained by keeping only the zero-order term of the effective Hamiltonian $W_{\text{eff}(0)}(t) = W^{(0)}(t)$ in the transformed TDSE (10). In other words, due to condition (21) one neglects of-diagonal terms $W^{(n-m)}(t)$ with $n \neq m$ which describe the coupling between different Floquet bands in Eq. (19), as illustrated in Fig. 1. This is equivalent to the time averaging of the transformed

Hamiltonian $W(\omega t + \theta, t)$ over fast oscillations. Consequently, the evolution operator (15) can be replaced by the effective operator for the adiabatic dynamics, $U(t, t_0) \approx U_{\text{eff}(0)}(t, t_0)$, with

$$U_{\text{eff}(0)}(t, t_0) = \mathcal{T} \exp\left[-\frac{i}{\hbar}\int_{t_0}^{t} W^{(0)}(t')dt'\right], \qquad (22)$$

where the time ordering $\mathcal{T}$ is needed if the effective Hamiltonian $W^{(0)}(t)$ does not commute with itself at different times, $[W^{(0)}(t'), W^{(0)}(t'')] \neq 0$.

B. Non-Abelian geometric phases

Calling on Eq. (12) for $W(\omega t + \theta, t)$, the adiabatic evolution operator (22) can be represented as

$$U_{\text{eff}(0)}(t, t_0) = \mathcal{T} \exp\left[-\frac{i}{\hbar}\int_{t_0}^{t} A_{\mu}^{(0)}(\boldsymbol{\lambda}(t'))d\lambda_{\mu}(t')\right]$$
$$= \exp\left[i\Gamma(t, t_0)\right], \qquad (23)$$

where

$$A_{\mu}^{(0)}(\boldsymbol{\lambda}) = \frac{1}{2\pi}\int_{0}^{2\pi} A_{\mu}(\theta', \boldsymbol{\lambda})d\theta' \qquad (24)$$

is the zero-frequency Fourier component of $A_{\mu}(\omega t + \theta, \boldsymbol{\lambda})$.

The evolution operator $U_{\text{eff}(0)}(t, t_0)$ does not depend on the speed of the change of the parameters $\boldsymbol{\lambda} = \boldsymbol{\lambda}(t)$. The operator $U_{\text{eff}(0)}(t, t_0)$ is defined exclusively by the trajectory along which the parameters $\boldsymbol{\lambda} = \{\lambda_{\mu}\}$ evolve. In particular, if the slowly varying parameters $\boldsymbol{\lambda} = \boldsymbol{\lambda}(t)$ undergo a cyclic evolution and return to their original values, $\boldsymbol{\lambda}(t) = \boldsymbol{\lambda}(t_0)$, the operator $U_{\text{eff}(0)}(t, t_0) = \exp(i\Gamma)$ is determined by the geometry of such a closed trajectory. Therefore the operator Γ featured in the exponent describes the geometric phase acquired during the cyclic evolution. When the parameters $\boldsymbol{\lambda} = \{\lambda_{\mu}\}$ complete two consecutive closed loop trajectories, the evolution of the system is described by the product of two geometric phase factors, $\exp(i\Gamma_1)$ and $\exp(i\Gamma_2)$, corresponding to each closed loop. If the two factors do not commute, one arrives at non-Abelian geometric phases. Thus the present work extends the previous studies of non-Abelian geometric phases [68–70] to the periodically driven (Floquet) system. In particular, non-Abelian geometric phases are formed for a spin in an oscillating magnetic field with a slowly changing direction, as we will see in Sec. V.

It is noteworthy that the non-Abelian geometric phases emerge because the system is subjected to the periodic driving. The periodic driving provides degenerate manifolds of Floquet states (shown by horizontal lines in Fig. 1) if one neglects $W(\omega t + \theta, t)$ appearing due to the slow changes of the driving. Such a degeneracy of the Floquet bands is related to the fact that the instantaneous eigenenergies of the Hamiltonian (1) average to zero over the driving period. The slow change of the driving induces the non-Abelian geometric phase factors represented by $W^{(0)}(t) = \dot{\lambda}_{\mu}A_{\mu}^{(0)}(\boldsymbol{\lambda})$, and which in turn describe the coupling within a degenerate Floquet band in the evolution operator given by Eqs. (22) or (23).

We are interested mostly in a situation where the periodically driven quantum system is characterized by a Hilbert space of finite dimension. In that case the number of the

Floquet states within each degenerate Floquet band n (represented by horizontal lines in Fig. 1) equals the number of the basis state vectors $|\alpha\rangle$, i.e., to the dimension of the Hilbert space $\mathscr{H}$. In particular, for the spin in the magnetic field described by Eq. (4), the Hilbert space $\mathscr{H}$ spans all spin projection states with quantum numbers $m_F = -f_F, -(f_F - 1), \ldots, +f_F$. The number of degenerate states then equals $2f_F + 1$ for each Floquet band, where f_F is the spin quantum number.

C. Return to the original representation and Floquet states

Returning to the original representation

$$|\phi(t)\rangle \equiv |\phi(\omega t + \theta, t)\rangle = R(\omega t + \theta, \boldsymbol{\lambda}(t))|\psi(t)\rangle, \qquad (25)$$

the adiabatic evolution of the state vector is given by

$$\begin{aligned}|\phi(\omega t + \theta, t)\rangle \\ = e^{-iS(\omega t + \theta, t)}U_{\text{eff}(0)}(t, t_0)e^{iS(\omega t_0 + \theta, t_0)}|\phi(t_0)\rangle,\end{aligned} \qquad (26)$$

where the oscillating Hermitian operator

$$S(\omega t + \theta, t) = \frac{\mathcal{F}(\omega t + \theta)}{\hbar\omega}V(\boldsymbol{\lambda}(t)) \qquad (27)$$

describes the fast micromotion of the state vector (26) due to the periodic driving. The solution $|\phi(\omega t + \theta, t)\rangle$ is thus 2π periodic with respect to the first variable $\omega t + \theta$. Additionally, $|\phi(\omega t + \theta, t)\rangle$ slowly changes with respect to the second variable t due to the temporal dependence of the operator $V(\boldsymbol{\lambda}(t))$ determining $S(\omega t + \theta, t)$ and $U_{\text{eff}(0)}(t, t_0)$.

If there is no periodic driving at the initial time, $V(\boldsymbol{\lambda}(t_0)) = 0$, and the driving is ramped up slowly afterwards, the evolution is not affected by micromotion due to the ramping of the periodic driving: $S(\omega t_0 + \theta, t_0) = 0$. If additionally the periodic perturbation is ramped down slowly before the final time t, there is no contribution due to the micromotion at the final time either, i.e., $S(\omega t + \theta, t) = 0$, and Eq. (26) reduces to

$$|\phi(\omega t + \theta, t)\rangle = U_{\text{eff}(0)}(t, t_0)|\phi(t_0)\rangle. \qquad (28)$$

In this way, if the driving is ramped up and down slowly, the micromotion does not contribute to the overall adiabatic evolution of the state vector $|\phi(\omega t + \theta, t)\rangle$. The evolution is determined exclusively by the operator $U_{\text{eff}(0)}(t, t_0)$ containing the non-Abelian geometric phases. This can be used for precisely controlling the dynamics of the quantum system, such as for manipulating of qubits. A specific sequence of ramping up and down of the driving will be discussed in Sec. V E for a spin in the oscillating magnetic field.

For purely periodic driving where $\boldsymbol{\lambda}(t) = \boldsymbol{\lambda}$ is constant, the operator $V(\boldsymbol{\lambda})$ is not changing, so that $U(t, t_0) = U_{\text{eff}(0)}(t, t_0) = 1$ and $S(\omega t + \theta, t) = S(\omega t + \theta)$. In that case Eq. (26) becomes an exact Floquet solution which does not have an additional slow temporal dependence: $|\phi(\omega t + \theta, t)\rangle \equiv |\phi(\omega t + \theta)\rangle$. By taking a set of states $|\phi(t_0)\rangle = |\alpha\rangle$ which form an orthonormal basis in the Hilbert space $\mathscr{H}$, one arrives at the corresponding set of the Floquet solutions:

$$|\phi_{\alpha}(\omega t + \theta)\rangle = e^{-iS(\omega t + \theta)}e^{iS(\omega t_0 + \theta)}|\alpha\rangle. \qquad (29)$$

The solutions (29) are strictly periodic $|\phi_\alpha(\omega t + \theta + 2\pi)\rangle = |\phi_\alpha(\omega t + \theta)\rangle$ and thus satisfy the Floquet theorem [71] with zero quasienergies (modulus the driving energy $\hbar\omega$) for any initial state $|\alpha\rangle$.

IV. ANALYSIS OF OPERATOR $W(\omega t + \theta, t)$

A. General equations

It is convenient to define a variable

$$c = c(\omega t + \theta) = \frac{\mathcal{F}(\omega t + \theta)}{\hbar\omega} \tag{30}$$

and treat $W(\omega t + \theta, t) = \tilde{W}(c, t)$ as a function of c and the slow time t. Differentiating $\tilde{W}(c, t)$ given by Eqs. (11), (7), and (30) with respect to c, one arrives at the following equation (see Appendix A):

$$\frac{\partial \tilde{W}}{\partial c} = -\hbar \dot{V} + i[V, \tilde{W}] \tag{31}$$

subject to the initial condition

$$\tilde{W}(c, t) = 0 \quad \text{for} \quad c = 0. \tag{32}$$

Here we write the full time derivative $\dot{V}$ rather than the partial derivative $\partial V/\partial t$, because the slowly changing operator $V = V(\boldsymbol{\lambda}(t))$ does not depend on c.

The solution to Eq. (31) can be expanded in powers of c, giving

$$\tilde{W}(c, t) = i\hbar \left\{ \frac{ic}{1!} \dot{V} + \frac{(ic)^2}{2!} [V, \dot{V}] \right.$$
$$\left. + \frac{(ic)^3}{3!} [V, [V, \dot{V}]] + \cdots \right\}. \tag{33}$$

B. Weak driving

Let us now consider the weak driving where Eq. (6) holds, and thus it is sufficient to keep the leading terms of the expansion (33). The first term in Eq. (33) proportional to $c = \mathcal{F}(\omega t + \theta)/\hbar\omega$ does not have the zero-frequency Fourier component and thus does not contribute to the effective Floquet Hamiltonian $W_{\text{eff}(0)} = W^{(0)}(t)$. Yet this term provides the leading contribution to the Fourier components $W^{(m)}(t) \approx i\dot{V} f^{(m)}/m\omega$ with $m \neq 0$. Thus the adiabatic condition (21) takes the form

$$|\dot{V}_{\alpha\beta}| f^{(m)} \ll m\hbar\omega^2 \quad \text{for} \quad m \neq 0 \tag{34}$$

in the case of the weak driving.

The second term in the expansion (33) yields the effective Hamiltonian for the weak driving:

$$W_{\text{eff}(0)}(t) = W^{(0)}(t) \approx \frac{-ip}{2\hbar\omega^2} [V(t), \dot{V}(t)], \tag{35}$$

with

$$p = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{F}^2(\theta') d\theta'. \tag{36}$$

For a harmonic driving, one has $f(\theta) = \cos\theta$ and $\mathcal{F}(\theta) = \sin\theta$, giving $p = 1/2$. In that case the effective Hamiltonian (35) coincides with Eq. (38) of Ref. [67] obtained in the

second order of the high-frequency expansion of the effective Hamiltonian in the original representation.

Generally it is not possible to obtain simple analytical expressions for the Floquet effective Hamiltonian $W_{\text{eff}(0)}(t) = W^{(0)}(t)$, similar to Eq. (35), beyond the weak driving regime. Yet such expressions can be obtained for specific models, such as for the spin in the oscillating magnetic field. This will be considered in Sec. V A.

C. $V(t)$ commutes with itself at different times

If $V(t)$ commutes with itself at different times, then $[V, \dot{V}] = 0$, so only the first term remains in the expansion (33), giving

$$W(\omega t + \theta, t) = -\mathcal{F}(\omega t + \theta)\dot{V}/\omega. \tag{37}$$

Since $\mathcal{F}(\omega t + \theta)$ averages to zero, the operator $W(\omega t + \theta, t)$ does not have the Fourier component $W^{(0)}(t)$, so the effective Hamiltonian is equal to zero, $W^{(0)}(t) = 0$. Therefore, in order to have a nontrivial evolution giving $W^{(0)}(t) \neq 0$, the operator $V(t)$ should not commute with itself at different times. For the spin in the fast oscillating magnetic field, this is the case if the direction of the magnetic field changes ($\mathbf{B} \times \dot{\mathbf{B}} \neq 0$), as we will see next.

V. SPIN IN OSCILLATING MAGNETIC FIELD

A. Explicit expression for operator W

For the spin in an oscillating magnetic field, described by the Hamiltonian (4), the operator W can be derived exactly for arbitrary strength of the periodic driving (see Appendix B):

$$W(\omega t + \theta, t) = -\frac{g_F \mathcal{F}}{\omega} \frac{(\mathbf{B} \cdot \dot{\mathbf{B}})\mathbf{B} \cdot \mathbf{F}}{B^2}$$
$$- \sin\left(\frac{B g_F \mathcal{F}}{\omega}\right) \frac{[(\mathbf{B} \times \dot{\mathbf{B}}) \times \mathbf{B}] \cdot \mathbf{F}}{B^3}$$
$$- \left[\cos\left(\frac{B g_F \mathcal{F}}{\omega}\right) - 1\right] \frac{(\mathbf{B} \times \dot{\mathbf{B}}) \cdot \mathbf{F}}{B^2} \tag{38}$$

with $\mathcal{F} = \mathcal{F}(\omega t + \theta)$ and $\mathbf{B} = \mathbf{B}(t)$. We will use this relation to analyze the dynamics of the spin in the oscillating magnetic field.

B. Effective Hamiltonian $W_{\text{eff}(0)}(t) = W^{(0)}(t)$

The first term in Eq. (38) averages to zero and does not contribute to the effective Hamiltonian $W^{(0)}(t)$. In what follows we will consider the harmonic driving where $f(\theta) = \cos\theta$ and hence $\mathcal{F}(\theta) = \sin\theta$. In that case the second term of Eq. (38) also averages to zero and thus does not contribute to $W^{(0)}(t)$. Therefore the effective Hamiltonian originates from the third term of Eq. (38) and is given by

$$W^{(0)}(t) = \frac{[1 - \mathcal{J}_0(g_F B/\omega)]}{B^2} \mathbf{F} \cdot (\mathbf{B} \times \dot{\mathbf{B}}), \tag{39}$$

where $\mathcal{J}_0(a) = \frac{1}{2\pi} \int_0^{2\pi} e^{ia\sin\theta} d\theta$ is the zero-order Bessel function, $B = |\mathbf{B}|$, and the time dependence of $\mathbf{B} = \mathbf{B}(t)$ is kept implicit. For $g_F B/\omega \ll 1$, Eq. (39) reduces to the previous
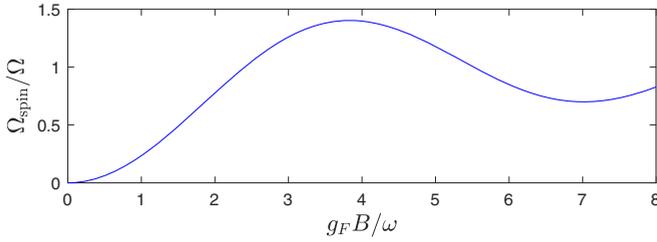
FIG. 2. Dependence of $\Omega_{\text{spin}}/\Omega$ on the ratio $g_F B/\omega$.

result [67] applicable for the weak driving:

$$W^{(0)}(t) \approx \frac{g_F^2}{4\omega^2} \mathbf{F} \cdot (\mathbf{B} \times \dot{\mathbf{B}}). \tag{40}$$

Introducing a unit vector along the magnetic field $\mathbf{b} = \mathbf{B}/B$, one has $\mathbf{b} \times \dot{\mathbf{b}} = \Omega \mathbf{n}$, where Ω can be interpreted as a frequency of the magnetic field rotation around an instantaneous rotation axis pointing along a unit vector $\mathbf{n}$. With these notations Eq. (39) takes the form

$$W^{(0)}(t) = \Omega[1 - \mathcal{J}_0(g_F B/\omega)]\mathbf{F} \cdot \mathbf{n}. \tag{41}$$

C. Adiabatic evolution

If the magnetic field rotates with a frequency Ω in a plane perpendicular to a fixed axis $\mathbf{n}$, the effective Hamiltonian given by Eq. (39) or (41) describes the spin rotation around $\mathbf{n}$ with a frequency

$$\Omega_{\text{spin}} = \Omega[1 - \mathcal{J}_0(a)], \text{ with } a = g_F B/\omega. \tag{42}$$

For $a \ll 1$, the frequency $\Omega_{\text{spin}} \approx \Omega a^2/4$ is much smaller than the rotation frequency Ω of the magnetic field. With an increase of a, the frequency Ω_{spin} increases. In particular, for $a \approx 2.4$ the Bessel function $\mathcal{J}_0(a)$ becomes zero, and the spin rotates with the same frequency as the magnetic field: $\Omega_{\text{spin}} = \Omega$ (see Fig. 2). By further increasing a, one arrives at a regime where $\mathcal{J}_0(a) < 0$, and the frequency Ω_{spin} exceeds Ω. The maximum frequency of the spin rotation $\Omega_{\text{spin}} = 1.36\Omega$ is achieved for $a \approx 3.83$ where the Bessel function $\mathcal{J}_0(a)$ has its minimum, as illustrated in Fig. 2.

If the direction $\mathbf{n}$ of the rotation axis is changing, the Hamiltonian $W^{(0)}(t)$ does not commute with itself at different times, and the time ordering is needed in the effective evolution operator (22). Therefore the effective evolution of the spin is associated with non-Abelian (noncommuting) geometric phases. In the present situation, the magnetic field $\mathbf{B}(t)$ plays the role of the slowly varying $\boldsymbol{\lambda}(t)$ featured in Sec. III B, so the effective Hamiltonian (39) can be represented in terms of the non-Abelian vector potential $\mathbf{A}^{(0)}(t)$:

$$W^{(0)}(t) = \dot{\mathbf{B}} \cdot \mathbf{A}^{(0)}, \tag{43}$$

with

$$\mathbf{A}^{(0)} = \frac{[1 - \mathcal{J}_0(g_F B/\omega)]}{B^2}(\mathbf{B} \times \mathbf{F}). \tag{44}$$

The evolution operator $U_{\text{eff}(0)}(t, t_0)$ is then given by Eq. (23) with $\boldsymbol{\lambda}(t)$ replaced by $\mathbf{B}(t)$.

In particular, one can perform a cyclic anticlockwise rotation of the magnetic field $\mathbf{B}$ in a plane orthogonal to a fixed

unit vector $\mathbf{n}$ without changing the modulus B. Using Eq. (43), the evolution operator (23) then reads

$$U_{\text{eff}(0)}^{(\mathbf{n})} = \exp\left[-\frac{i}{\hbar}\gamma \mathbf{F} \cdot \mathbf{n}\right], \ \gamma = 2\pi[1 - \mathcal{J}_0(g_F B/\omega)]. \tag{45}$$

The operator $\gamma \mathbf{F} \cdot \mathbf{n}/\hbar$ provides the geometric phase γm_F for the spin with the projection m_F along the rotation axis $\mathbf{n}$. For weak driving ($g_F B/\omega \ll 1$) the geometric phase γm_F is much smaller than unity, and the magnetic field $\mathbf{B}$ has to complete many rotation cycles to accumulate a considerable geometric phase [67]. On the other hand, if $g_F B$ is comparable with ω, a sizable geometric phase is acquired during a single cycle. Therefore two consecutive rotations along nonparallel axes $\mathbf{n}$ and $\mathbf{n}'$ do not commute, $[U_{\text{eff}(0)}^{(\mathbf{n})}, U_{\text{eff}(0)}^{(\mathbf{n}')}] \neq 0$, and the corresponding geometric phases $\gamma \mathbf{F} \cdot \mathbf{n}/\hbar$ and $\gamma \mathbf{F} \cdot \mathbf{n}'/\hbar$ are non-Abelian.

Previously, Berry analyzed a spin that adiabatically follows a slowly changing magnetic field [72]. After the magnetic field vector completes a closed loop trajectory and returns to its initial value, the state vector for the spin acquires a geometric (Berry) phase factor. Such a phase factor belongs to the Abelian group $U(1)$. For the periodically driven spin considered here, the adiabatic evolution of the state vector in the degenerate Floquet manifold is described by the geometric phase operator $U_{\text{eff}(0)}(t, t_0) = \exp(i\Gamma)$ belonging to the non-Abelian group $SU(2)$. Therefore the periodic driving enriches the system. Note that in the present situation the non-Abelian Floquet geometric phases appear by adiabatically eliminating other Floquet bands rather than by eliminating other states of the physical Hilbert space $\mathscr{H}$, as is the case for nondriven systems [68–70].

D. Adiabatic condition and micromotion

The previous analysis of the spin in the oscillating magnetic field [67] relies on the high-frequency assumption for the magnetic field amplitude, $g_F f_F B(t) \ll \omega$, and for its changes. Now we require only that $\mathbf{B}(t)$ changes sufficiently slowly, so that the adiabatic condition (21) holds for W given by Eq. (38). The general expression for the adiabatic condition given by Eqs. (21) and (38) is quite cumbersome. Yet if

$$g_F f_F |\dot{\mathbf{B}}(t)|/\omega \ll \omega, \tag{46}$$

the adiabatic condition is fulfilled for any strength of the magnetic field. In other words, if (46) holds, then the adiabatic condition (21) is fulfilled but not vice versa.

The micromotion is described by the Hermitian operator $S(\omega t + \theta, t)$ entering the full evolution operator in Eq. (26). Using Eqs. (4) and (27), the micromotion operator reads for the spin in the magnetic field

$$S(\omega t + \theta, t) = \frac{g_F}{\hbar\omega}\mathbf{F} \cdot \mathbf{B}(t) \sin(\omega t + \theta). \tag{47}$$

The micromotion increases with increasing the magnetic field strength and becomes substantial when the high-frequency condition ($g_F f_F B(t) \ll \omega$) no longer holds. Yet, if the magnetic field is ramped up and down slowly, the micromotion does affect the overall evolution of the system, as was generally shown in Eq. (28) and will be discussed in more detail next.
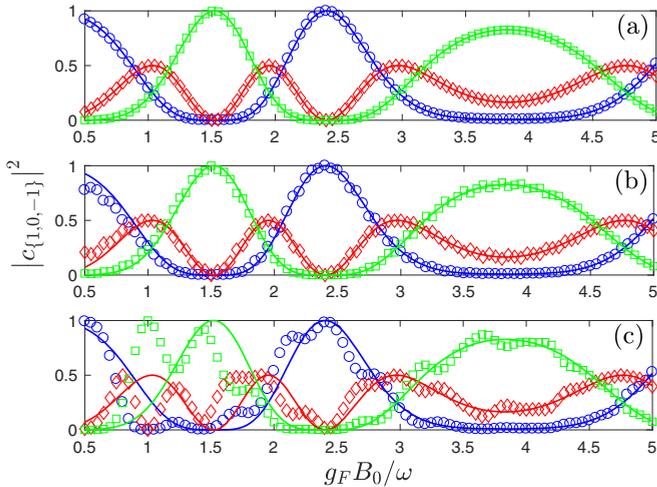
FIG. 3. Comparison of the analytical (solid lines) and exactly calculated (symbols) evolution for the spin 1 ($f_F = 1$). Here $|c_{m'_F}|^2$ are the probabilities for the spin projection along the z axis to be $m'_F = 0, \pm 1$ at the final time $t = t_3$. At the initial time $t = t_0$ the spin is along the z axis ($m_F = 1$). Blue circles, red diamonds, and green squares correspond to $|c_1|^2$, $|c_0|^2$, and $|c_{-1}|^2$, respectively. The effective evolution described by the operator $U_{\text{eff}(0)}(t_2, t_1) = U_{\text{eff}(0)}^{(\mathbf{e}_y)}$ provides the following probabilities: $|c_{\{1,0,-1\}}|^2 = \{\cos^4(\gamma/2), \sin^2(\gamma)/2, \sin^4(\gamma/2)\}$ plotted by the solid lines, with γ defined in Eq. (45). The ratio $\rho = \Omega g_F B_0/\omega^2$ equals $\rho = 0.1$, $\rho = 0.3$, and $\rho = 1$ in panels (a), (b), and (c), respectively.

E. Specific sequence of the magnetic field

The non-Abelian geometric phases can be measured using, for example, the following sequence for the oscillating magnetic field. At the initial time $t = t_0$ the magnetic field is zero ($\mathbf{B}(t_0) \to 0$) and is ramped up smoothly afterwards, so there is no contribution by the micromotion due to switching on the magnetic field: $S(\theta + \omega t_0, t_0) = 0$. For $t_0 < t < t_1$ the amplitude of magnetic field increases from zero to a steady-state value $\mathbf{B}(t) = B_0 \mathbf{e}_z$ without changing its direction. Therefore the effective Hamiltonian $W_{\text{eff}(0)} = W^{(0)}$ is zero in this stage, giving $U_{\text{eff}(0)}(t_1, t_0) = 1$. During the subsequent evolution at $t_1 < t < t_2$ the magnetic field changes its direction while keeping constant the modulus B. For example, the magnetic field can undergo two consecutive cycles of rotation, around the y and x axes, described by noncommuting unitary operators $U_{\text{eff}(0)}^{(\mathbf{e}_y)}$ and $U_{\text{eff}(0)}^{(\mathbf{e}_x)}$ given by Eq. (45), and the effective evolution operator reads $U_{\text{eff}(0)}(t_2, t_1) = U_{\text{eff}(0)}^{(\mathbf{e}_x)} U_{\text{eff}(0)}^{(\mathbf{e}_y)}$. Note that for $t_1 < t < t_0$ the magnetic field strength can be considerable, and the high-frequency condition $g_F f_F B(t) \ll \omega$ does not necessarily hold. Therefore the evolution operator $U_{\text{eff}(0)}(t_2, t_1)$ can significantly alter the state vector of the system. Finally, for $t_2 < t < t_3$ the magnetic field is ramped down to zero without changing its direction, giving $U_{\text{eff}(0)}(t_3, t_2) = 1$ and $S(\theta + \omega t_3, t_3) = 0$. In this way, the full evolution of the state vector from $t = t_0$ to $t = t_3$ is given by Eq. (28) with t replaced by t_3. The evolution of the state vector is thus described exclusively by the operator $U_{\text{eff}(0)}(t_2, t_1)$, which does not depend on the details of the ramping up and down of the magnetic field.

In Fig. 3 we have checked the validity of the description of the evolution of the system in terms of the effective evolution operator $U_{\text{eff}(0)}(t_2, t_1)$. We have plotted the exact and effective evolution of the spin 1 system ($f_F = 1$) from $t = t_0$ to $t = t_3$ for different angular frequencies Ω of rotation of the magnetic field direction during the second stage where $t_1 < t < t_2$. We have considered the case where the magnetic field completes a single cycle of rotation around the y axis from $t = t_1$ to $t = t_2$, so that $U_{\text{eff}(0)}(t_2, t_1) = U_{\text{eff}(0)}^{(\mathbf{e}_y)}$. The angular frequency Ω is chosen such that the ratio $\rho = \Omega g_F B_0/\omega^2$ is not changing in the same plot. The exact and analytical results agree well if the rotation is sufficiently slow, $\rho \ll 1$, and thus the adiabatic condition (46) holds.

VI. CONCLUDING REMARKS

We have considered the evolution of a periodically driven quantum system governed by the Hamiltonian $H(\omega t + \theta, t)$, which is the product of a slowly varying Hermitian operator $V(\boldsymbol{\lambda}(t))$ and a fast oscillating periodic function $f(\omega t + \theta)$ with zero average. The analysis does not rely on the high-frequency approximation [67] for the original Hamiltonian $H(\omega t + \theta, t)$, so the driving frequency ω can be both larger or smaller than the matrix elements of the slowly changing operator $V(\boldsymbol{\lambda}(t))/\hbar$. We have shown that the adiabatic evolution of the system within a degenerate Floquet band is accompanied by the non-Abelian (noncommuting) Floquet geometric phases, which can be significant even after completing a single cycle of the slow variable $\boldsymbol{\lambda} = \boldsymbol{\lambda}(t)$. On the other hand, for the weak driving the geometric phases acquired during a cyclic evolution of the slow variable are small, and the slow variable should complete many cycles to accumulate considerable geometric phases [67].

Without the periodic driving $f(\omega t + \theta)$ the spin adiabatically follows the slowly changing magnetic field, and the adiabatic elimination of other spin states provides the Berry phase factor [72], which belongs to the Abelian group $U(1)$. The periodic driving enriches the system, and the non-Abelian geometric phases appear by adiabatically eliminating other Floquet bands rather than by eliminating other states of the physical Hilbert space $\mathcal{H}$, as is the case for nondriven systems [68–70]. In the latter nondriven systems non-Abelian geometric phases can be formed if there is a manifold of degenerate physical states well separated from other states, such as a pair of degenerate dark states in the tripod atom-light coupling scheme [73–77] or a pair of degenerate spin-up and spin-down states emerging in the nuclear quadrupole resonance [70] and for diatomic molecules [69,78]. Note also that the previous studies of the Floquet adiabatic perturbation theory [9] dealt with the nondegenerate Floquet states, so the emerging geometric phases are Abelian.

A distinctive feature of the present periodically driven system is that the Floquet eigen-energies are fully degenerate within individual Floquet bands even if the eigenenergies of the slowly varying part of the original Hamiltonian $V(\boldsymbol{\lambda}(t))$ are not degenerate. Therefore the non-Abelian geometric phases emerge in a very straightforward way, and no degeneracy of the physical states is needed. Furthermore, the individual Floquet bands are characterized by zero energy

(modulus the driving frequency), so there are no unwanted dynamical phases accompanying the non-Abelian geometric phases during the adiabatic evolution of the system within individual Floquet bands. This is because the dynamical phases average to zero over an oscillation period.

The adiabatic evolution of periodically driven systems is generally accompanied by micromotion. Yet the effects of the micromotion can be avoided if the periodic driving is ramped up slowly at the initial stage and subsequently ramped down slowly at the final stage. The dynamics of the state vector given by Eq. (28) is then represented exclusively by the operator $U_{\mathrm{eff}(0)}(t, t_0)$ describing the non-Abelian geometric phases emerging for the adiabatic evolution of the system within a degenerate Floquet band. The geometric phases are determined by the trajectory of the slowly varying parameters $\boldsymbol{\lambda}$ rather than by the speed at which these parameters change. This can be used for precisely controlling the evolution of quantum systems, in particular for realization of fault-tolerant quantum gates [79].

ACKNOWLEDGMENTS

APPENDIX A: EQUATION FOR OPERATOR W

Differentiating $W(\omega t + \theta, t) = \tilde{W}(c, t)$ given by Eqs. (11), (7), and (30) with respect to c for fixed slow time t, one has

$$\frac{\partial \tilde{W}}{\partial c} = \hbar V R^{\dagger} \frac{\partial}{\partial t} R - \hbar R^{\dagger} \frac{\partial}{\partial t} (RV). \tag{A1}$$

Since $\hbar R^{\dagger} \frac{\partial}{\partial t}(RV) = \hbar \dot{V} + i\tilde{W}V$, Eq. (A1) yields the differential equation for $\tilde{W}$:

$$\frac{\partial \tilde{W}}{\partial c} = -\hbar \dot{V} + i[V, \tilde{W}] \tag{A2}$$

subject to the initial condition: $\tilde{W}(c, t) = 0$ for $c = 0$.

APPENDIX B: OPERATOR W FOR SPIN IN OSCILLATING MAGNETIC FIELD

Let us now find a solution to Eq. (A2) for a spin in an oscillating magnetic field. In that case the slowly varying part of the Hamiltonian is given by Eq. (4). We are looking for a solution in the form

$$\tilde{W}(c, t) = \mathbf{F} \cdot \mathbf{X}. \tag{B1}$$

Substituting Eq. (B1) into (A2), one arrives at the following equation for the vector $\mathbf{X} = \mathbf{X}(c, t)$:

$$\frac{\partial \mathbf{X}}{\partial c} = -\hbar g_F \dot{\mathbf{B}} - \hbar g_F \mathbf{B} \times \mathbf{X}, \tag{B2}$$

with the initial condition $\mathbf{X}(c, t) = 0$ for $c = 0$. A solution to this equation is

$$\begin{aligned}
\mathbf{X}(c, t) = &- c\hbar g_F \frac{(\mathbf{B} \cdot \dot{\mathbf{B}})\mathbf{B}}{B^2} \\
&- \sin(c\hbar g_F B) \frac{(\mathbf{B} \times \dot{\mathbf{B}}) \times \mathbf{B}}{B^3} \\
&- [\cos(c\hbar g_F B) - 1] \frac{\mathbf{B} \times \dot{\mathbf{B}}}{B^2}.
\end{aligned} \tag{B3}$$

Equations (B1), (B3), and (30) provide the explicit expression for $\tilde{W}(c, t) = W(\omega t + \theta, t)$ presented by Eq. (38).

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