Real measurements and the quantum Zeno effect

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In 1977, Mishra and Sudarshan [J. Math. Phys. **18**, 756 (1977)] showed that an unstable particle would never be found decayed while it was continuously observed. They called this effect the quantum Zeno effect (or paradox). Later it was realized that the frequent measurements could also accelerate the decay (quantum anti-Zeno effect). In this paper, we investigate the quantum Zeno effect using the definite model of the measurement. We take into account the finite duration and the finite accuracy of the measurement. A general equation for the jump probability during the measurement is derived. We find that the measurements can cause inhibition (quantum Zeno effect) or acceleration (quantum anti-Zeno effect) of the evolution, depending on the strength of the interaction with the measuring device and on the properties of the system. However, the evolution cannot be fully stopped.

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I. INTRODUCTION

Theory of measurements has a special status in quantum mechanics. Unlike classical mechanics, in quantum mechanics it cannot be assumed that the effect of the measurement on the system can be made arbitrarily small. It is necessary to supplement quantum theory with additional postulates, describing the measurement. One such additional postulate is von Neumann's state reduction (or projection) postulate [1]. The essential peculiarity of this postulate is its nonunitary character. However, this postulate refers only to an ideal measurement, which is instantaneous and arbitrarily accurate. Real measurements are described by the projection postulate only roughly.

The important consequence of von Neumann's projection postulate is the quantum Zeno effect. In quantum mechanics, short-time behavior of the nondecay probability of unstable particles is not exponential but quadratic [2]. This deviation from the exponential decay has been observed by Wilkinson *et al.* [3]. In 1977, Mishra and Sudarshan [4] showed that this behavior when combined with the quantum theory of measurement, based on the assumption of the collapse of the wave function, led to a very surprising conclusion: frequent observations slowed down the decay. An unstable particle would never decay when continuously observed. Mishra and Sudarshan have called this effect the quantum Zeno paradox or effect. The effect is so called in allusion to the paradox stated by the Greek philosopher Zeno (or Zenon) of Elea. The very first analysis does not take into account the actual mechanism of the measurement process involved, but it is based on an alternating sequence of unitary evolution and a collapse of the wave function. The Zeno effect has been experimentally proved [5] in a repeatedly measured two-level system undergoing Rabi oscillations. The outcome of this experiment has also been explained without the collapse hypothesis [6–8].

Later it was realized that the repeated measurements could not only slow the quantum dynamics, but the quantum process may be accelerated by frequent measurements as well [9–15]. This effect was called a quantum anti-Zeno effect by Kaulakys and Gontis [10], who argued that frequent interrogations may destroy the quantum localization effect in chaotic systems. An effect, analogous to the quantum anti-Zeno effect, has been obtained in a computational study involving barrier penetration, too [16]. Recently, an analysis of the acceleration of a chemical reaction due to the quantum anti-Zeno effect has been presented in Ref. [17].

Although great progress in the investigation of the quantum Zeno effect has been made, this effect is not completely understood as yet. In the analysis of the quantum Zeno effect, the finite duration of the measurement becomes important, therefore the projection postulate is not sufficient to solve this problem. The complete analysis of the Zeno effect requires a more precise model of measurement than the projection postulate.

The purpose of this paper is to consider such a model of the measurement. The model describes a measurement of the finite duration and finite accuracy. Although the model used does not describe the irreversible process, it leads, however, to the correct correlation between the states of the measured system and the measuring apparatus.

Due to the finite duration of the measurement, it is impossible to consider infinitely frequent measurements, as in Ref. [4]. The highest frequency of the measurements is achieved when the measurements are performed one after another, without the period of the measurement-free evolution between two successive measurements. In this paper, we consider such a sequence of measurements. Our goal is to check whether this sequence of measurements can change the evolution of the system and to verify the predictions of the quantum Zeno effect.

The work is organized as follows. In Sec. II, we present the model of the measurement. A simple case is considered in Sec. III in order to determine the requirements for the duration of the measurement. In Sec. IV, we derived a general formula for the probability of the jump into another level during the measurement. The effect of repeated measurements on the system with a discrete spectrum is investigated in Sec. V. The decaying system is considered in Sec. VI. Section VII summarizes our findings.

II. MODEL OF THE MEASUREMENTS

We consider a system that consists of two parts. The first part of the system has the discrete energy spectrum. The

Hamiltonian of this part is $\hat{H}_0$. The other part of the system is represented by Hamiltonian $\hat{H}_1$. Hamiltonian $\hat{H}_1$ commutes with $\hat{H}_0$. In a particular case, the second part can be absent and $\hat{H}_1$ can be zero. The operator $\hat{V}(t)$ causes the jumps between different energy levels of $\hat{H}_0$. Therefore, the full Hamiltonian of the system is equal to $\hat{H}_S=\hat{H}_0+\hat{H}_1+\hat{V}(t)$. The example of such a system is an atom with the Hamiltonian $\hat{H}_0$ interacting with the electromagnetic field, represented by $\hat{H}_1$.

We will measure in which eigenstate of the Hamiltonian $\hat{H}_0$ the system is. The measurement is performed by coupling the system with the detector. The full Hamiltonian of the system and the detector is equal to

$$\hat{H}=\hat{H}_S+\hat{H}_D+\hat{H}_I, \tag{1}$$

where $\hat{H}_D$ is the Hamiltonian of the detector and $\hat{H}_I$ represents the interaction between the detector and the system. We choose the operator $\hat{H}_I$ in the form

$$\hat{H}_I=\lambda\hat{q}\hat{H}_0, \tag{2}$$

where $\hat{q}$ is the operator acting in the Hilbert space of the detector and the parameter λ describes the strength of the interaction. This system-detector interaction is that considered by von Neumann [1] and in Refs. [18–22]. In order to obtain a sensible measurement, the parameter λ must be large. We require a continuous spectrum of operator $\hat{q}$. For simplicity, we can consider the quantity q as the coordinate of the detector.

The measurement begins at time moment t_0. At the beginning of the interaction with the detector, the detector is in the pure state $|\Phi\rangle$. The full density matrix of the system and detector is $\hat{\rho}(t_0)=\hat{\rho}_S(t_0)\otimes|\Phi\rangle\langle\Phi|$, where $\hat{\rho}_S(t_0)$ is the density matrix of the system. The duration of the measurement is τ. After the measurement, the density matrix of the system is $\hat{\rho}_S(\tau+t_0)=\mathrm{Tr}_D\{\hat{U}(\tau+t_0)[\hat{\rho}_S(t_0)\otimes|\Phi\rangle\langle\Phi|]\hat{U}^\dagger(\tau+t_0)\}$ and the density matrix of the detector is $\hat{\rho}_D(\tau+t_0)=\mathrm{Tr}_S\{\hat{U}(\tau+t_0)[\hat{\rho}_S(t_0)\otimes|\Phi\rangle\langle\Phi|]\hat{U}^\dagger(\tau+t_0)\}$, where $\hat{U}(t)$ is the evolution operator of the system and detector, obeying the equation

$$i\hbar\frac{\partial}{\partial t}\hat{U}(t)=\hat{H}(t)\hat{U}(t) \tag{3}$$

with the initial condition $\hat{U}(t_0)=1$.

Since the initial density matrix is chosen in a factorizable form, the density matrix of the system after the interaction depends linearly on the density matrix of the system before the interaction. We can represent this fact by the equality

$$\hat{\rho}_S(\tau+t_0)=S(\tau,t_0)\hat{\rho}_S(t_0), \tag{4}$$

where $S(\tau,t_0)$ is the superoperator acting on the density matrices of the system. If the vectors $|n\rangle$ form the complete basis in the Hilbert space of the system, we can rewrite Eq. (4) in the form

$$\rho_S(\tau+t_0)_{pr}=S(\tau,t_0)^{nm}_{pr}\rho_S(t_0)_{nm}, \tag{5}$$

where the sum over the repeating indices is supposed. The matrix elements of the superoperator are

$$S(\tau,t_0)^{nm}_{pr}=\mathrm{Tr}_D\{\langle p|\hat{U}(\tau+t_0)(|n\rangle\langle m|\otimes|\Phi\rangle\langle\Phi|)\hat{U}^\dagger \times(\tau+t_0)|r\rangle\}. \tag{6}$$

Due to the finite duration of the measurement, it is impossible to realize the infinitely frequent measurements. The highest frequency of the measurements is achieved when the measurements are performed one after another without the period of the measurement-free evolution between two successive measurements. Therefore, we model a continuous measurement by the subsequent measurements of the finite duration and finite accuracy. After N measurements, the density matrix of the system is

$$\hat{\rho}_S(N\tau)=S(\tau,(N-1)\tau)\cdots S(\tau,\tau)S(\tau,0)\hat{\rho}_S(0). \tag{7}$$

Further, for simplicity we will neglect the Hamiltonian of the detector. After this assumption, the evolution operator is equal to $\hat{U}(t,1+\lambda\hat{q})$, where the operator $\hat{U}(t,\xi)$ obeys the equation

$$i\hbar\frac{\partial}{\partial t}\hat{U}(t,\xi)=[\xi\hat{H}_0+\hat{H}_1+\hat{V}(t)]\hat{U}(t,\xi) \tag{8}$$

with the initial condition $\hat{U}(t_0,\xi)=1$. Then the superoperator $S(\tau,t_0)$ is

$$S(\tau,t_0)^{nm}_{pr}=\int dq|\langle q|\Phi\rangle|^2\langle p|\hat{U}(\tau+t_0,1+\lambda q)|n\rangle\langle m|\hat{U}^\dagger \times(\tau+t_0,1+\lambda q)|r\rangle. \tag{9}$$

III. MEASUREMENT OF THE UNPERTURBED SYSTEM

In order to estimate the necessary duration of the single measurement, it is convenient to consider the case when the operator $\hat{V}=0$. In such a case, the description of the evolution is simpler. The measurement of this kind occurs also when the influence of the perturbation operator $\hat{V}$ is small in comparison with the interaction between the system and the detector and, therefore, the operator $\hat{V}$ can be neglected.

We can choose the basis $|n\alpha\rangle$ common for the operators $\hat{H}_0$ and $\hat{H}_1$,

$$\hat{H}_0|n\alpha\rangle=E_n|n\alpha\rangle, \tag{10}$$

$$\hat{H}_1|n\alpha\rangle=E_1(n,\alpha)|n\alpha\rangle, \tag{11}$$

where n is the number of eigenvalues of the Hamiltonian $\hat{H}_0$ and α represents the remaining quantum numbers. Since the

Hamiltonian of the system does not depend on t, we will omit the parameter t_0 in this section. From Eq. (9), we obtain the superoperator $S(\tau)$ in the basis $|n\alpha\rangle$,

$$S(\tau)^{n\alpha_1,m\alpha_2}_{p\alpha_3,r\alpha_4}=\delta_{np}\delta_{mr}\delta(\alpha_1,\alpha_3)\delta(\alpha_2,\alpha_4)\exp(i\omega_{m\alpha_2,n\alpha_1}\tau)\times\int dq|\langle q|\Phi\rangle|^2\exp(i\lambda\omega_{mn}\tau q), \tag{12}$$

where

$$\omega_{mn}=\frac{1}{\hbar}(E_m-E_n), \tag{13}$$

$$\omega_{m\alpha_2,n\alpha_1}=\omega_{mn}+\frac{E_1(m,\alpha_2)-E_1(n,\alpha_1)}{\hbar}, \tag{14}$$

and $\delta(\cdot,\cdot)$ represent Kronecker's delta in a discrete case and Dirac's delta in a continuous case. Equation (12) can be rewritten using the correlation function

$$F(\nu)=\langle\Phi|\exp(i\nu\hat{q})|\Phi\rangle. \tag{15}$$

We can express this function as $F(\nu)=\int dq|\langle q|\Phi\rangle|^2\exp(i\nu q)=\int dp\langle\Phi|p\rangle\langle p-(\nu/\hbar)|\Phi\rangle$. Since vector $|\Phi\rangle$ is normalized, the function $F(\nu)$ tends to zero when $|\nu|$ increases. There exists a constant C such that the correlation function $|F(\nu)|$ is small if the variable $|\nu|>C$.

The equation for the superoperator $S(\tau)$ is

$$S(\tau)^{n\alpha_1,m\alpha_2}_{p\alpha_3,r\alpha_4}=\delta_{np}\delta_{mr}\delta(\alpha_1,\alpha_3)\delta(\alpha_2,\alpha_4)\times\exp(i\omega_{m\alpha_2,n\alpha_1}\tau)F(\lambda\tau\omega_{mn}). \tag{16}$$

Using Eqs. (5) and (16), we find that after the measurement, the nondiagonal elements of the density matrix of the system become small, since $F(\lambda\tau\omega_{mn})$ is small for $n\neq m$ when $\lambda\tau$ is large.

The density matrix of the detector is

$$\langle q|\hat{\rho}_D(\tau)|q_1\rangle=\langle q|\Phi\rangle\langle\Phi|q_1\rangle\mathrm{Tr}\{\hat{U}(\tau,1+\lambda q)\times\hat{\rho}_S(0)\hat{U}^\dagger(\tau,1+\lambda q_1)\}. \tag{17}$$

From Eqs. (8) and (17), we obtain

$$\langle q|\hat{\rho}_D(\tau)|q_1\rangle=\langle q|\Phi\rangle\langle\Phi|q_1\rangle\sum_n\exp(i\lambda\tau\omega_n(q_1-q))\times\sum_\alpha\langle n,\alpha|\hat{\rho}_S(0)|n,\alpha\rangle, \tag{18}$$

where

$$\omega_n=\frac{1}{\hbar}E_n. \tag{19}$$

The probability that the system is in the energy level n may be expressed as

$$P(n)=\sum_\alpha\langle n,\alpha|\hat{\rho}_S(0)|n,\alpha\rangle. \tag{20}$$

Introducing the state vectors of the detector

$$|\Phi_E\rangle=\exp\left(-\frac{i}{\hbar}\lambda\tau E\hat{q}\right)|\Phi\rangle, \tag{21}$$

we can express the density operator of the detector as

$$\hat{\rho}_D(\tau)=\sum_n|\Phi_{E_n}\rangle\langle\Phi_{E_n}|P(n). \tag{22}$$

The measurement is complete when the states $|\Phi_E\rangle$ are almost orthogonal. The different energies can be separated only when the overlap between the corresponding states $|\Phi_E\rangle$ is almost zero. The scalar product of the states $|\Phi_E\rangle$ with different energies E_1 and E_2 is

$$\langle\Phi_{E_1}|\Phi_{E_2}\rangle=F(\lambda\tau\omega_{12}). \tag{23}$$

The correlation function $|F(\nu)|$ is small when $|\nu|>C$. Therefore, we have the estimation for the error of the energy measurement ΔE as

$$\lambda\tau\Delta E\gtrsim\hbar C \tag{24}$$

and we obtain the expression for the necessary duration of the measurement

$$\tau\gtrsim\frac{\hbar}{\Lambda\Delta E}, \tag{25}$$

where

$$\Lambda=\frac{\lambda}{C}. \tag{26}$$

Since in our model the measurements are performed immediately one after the other, from Eq. (25) it follows that the rate of measurements is proportional to the strength of the interaction λ between the system and the measuring device.

IV. MEASUREMENT OF THE PERTURBED SYSTEM

The operator $\hat{V}(t)$ represents the perturbation of the unperturbed Hamiltonian $\hat{H}_0+\hat{H}_1$. We will take into account the influence of the operator $\hat{V}$ by the perturbation method, assuming that the strength of the interaction λ between the system and detector is large.

The operator $\hat{V}(t)$ in the interaction picture is

$$\tilde{V}(t,t_0,\xi)=\exp\left(\frac{i}{\hbar}(\xi\hat{H}_0+\hat{H}_1)t\right)\hat{V}(t+t_0)\times\exp\left(-\frac{i}{\hbar}(\xi\hat{H}_0+\hat{H}_1)t\right). \tag{27}$$

In the second-order approximation, the evolution operator is equal to

$$\hat{U}(\tau,t_0,\xi)\approx\exp\left(-\frac{i}{\hbar}(\xi\hat{H}_0+\hat{H}_1)\tau\right)\left\{1+\frac{1}{i\hbar}\int_0^\tau dt\,\tilde{V}(t,t_0,\xi)-\frac{1}{\hbar^2}\int_0^\tau dt_1\int_0^t dt_2\tilde{V}(t_1,t_0,\xi)\tilde{V}(t_2,t_0\xi)\right\}. \quad (28)$$

Using Eqs. (9) and (28), we can obtain the superoperator S in the second-order approximation, too. The expression for the matrix elements of the superoperator S is given in the Appendix [Eqs. (A1), (A2), and (A3)].

The probability of the jump from the level $|i\alpha\rangle$ to the level $|f\alpha_1\rangle$ during the measurement is $W(i\alpha\rightarrow f\alpha_1)=S(\tau,t_0)^{i\alpha,i\alpha}_{f\alpha_1,f\alpha_1}$. Using Eqs. (A1), (A2), and (A3), we obtain

$$W(i\alpha\rightarrow f\alpha_1)=\frac{1}{\hbar^2}\int_0^\tau dt_1\int_0^\tau dt_2F(\lambda\omega_{if}(t_2-t_1))\times V(t_1+t_0)_{f\alpha_1,i\alpha}V(t_2+t_0)_{i\alpha,f\alpha_1}\times\exp(i\omega_{i\alpha,f\alpha_1}(t_2-t_1)). \quad (29)$$

The expression for the jump probability can be further simplified if the operator $\hat{V}$ does not depend on t. We introduce the function

$$\Phi(t)_{f\alpha_1,i\alpha}=|V_{f\alpha_1,i\alpha}|^2\exp\left(\frac{i}{\hbar}[E_1(f,\alpha_1)-E_1(i,\alpha)]t\right). \quad (30)$$

Changing variables, we can rewrite the jump probability as

$$W(i\alpha\rightarrow f\alpha_1)=\frac{2}{\hbar^2}\mathrm{Re}\int_0^\tau dtF(\lambda\omega_{fi}t)\exp(i\omega_{fi}t)\times(\tau-t)\Phi(t)_{f\alpha_1,i\alpha}. \quad (31)$$

Introducing the Fourier transformation of $\Phi(t)_{f\alpha_1,i\alpha}$,

$$G(\omega)_{f\alpha_1,i\alpha}=\frac{1}{2\pi}\int_{-\infty}^{\infty}dt\,\Phi(t)_{f\alpha_1,i\alpha}\exp(-i\omega t), \quad (32)$$

and using Eq. (31), we obtain the equality

$$W(i\alpha\rightarrow f\alpha_1)=\frac{2\pi\tau}{\hbar^2}\int_{-\infty}^{\infty}d\omega\,G(\omega)_{f\alpha_1,i\alpha}P(\omega)_{if}, \quad (33)$$

where

$$P(\omega)_{if}=\frac{1}{\pi}\mathrm{Re}\int_0^\tau dtF(\lambda\omega_{if}t)\exp(i(\omega-\omega_{if})t)\left(1-\frac{t}{\tau}\right). \quad (34)$$

From Eq. (34), using the equality $F(0)=1$, we obtain

$$\int d\omega P(\omega)_{if}=1. \quad (35)$$

The quantity G is equal to

$$G(\omega)_{f\alpha_1,i\alpha}=\hbar|V_{f\alpha_1,i\alpha}|^2\delta(E_1(f,\alpha_1)-E_1(i,\alpha)-\hbar\omega). \quad (36)$$

We see that the quantity $G(\omega)$ characterizes the perturbation.

V. DISCRETE SPECTRUM

Let us consider the measurement effect on the system with the discrete spectrum. The Hamiltonian $\hat{H}_0$ of the system has a discrete spectrum, the operator $\hat{H}_1=0$, and the operator $\hat{V}(t)$ represents a perturbation resulting in the quantum jumps between the discrete states of the system $\hat{H}_0$.

For the separation of the energy levels, the error in the measurement should be smaller than the distance between the nearest energy levels of the system. It follows from this requirement and Eq. (25) that the measurement time $\tau\gtrsim 1/\Lambda\omega_{\min}$, where $\omega_{\min}$ is the smallest of the transition frequencies $|\omega_{if}|$.

When λ is large, then $|F(\lambda x)|$ is not very small only in the region $|x|<\Lambda^{-1}$. We can estimate the probability of the jump to the other energy level during the measurement, replacing $F(\nu)$ by $2C\delta(\nu)$ in Eq. (29). Then from Eq. (29) we obtain

$$W(i\alpha\rightarrow f\alpha_1)\approx\frac{2}{\hbar^2\Lambda|\omega_{if}|}\int_0^\tau dt|V(t+t_0)_{i\alpha,f\alpha_1}|^2. \quad (37)$$

We see that the probability of the jump is proportional to Λ^{-1}. Consequently, for large Λ, i.e., for the strong interaction with the detector, the jump probability is small. This fact represents the quantum Zeno effect. However, due to the finiteness of the interaction strength, the jump probability is not zero. After a sufficiently large number of measurements, the jump occurs. We can estimate the number of measurements N after which the system jumps into other energy levels from the equality $(2\tau/\hbar^2\Lambda|\omega_{\min}|)|V_{\max}|^2N\sim 1$, where $|V_{\max}|$ is the largest matrix element of the perturbation operator V. This estimation allows us to introduce the characteristic time, during which the evolution of the system is inhibited,

$$t_{\mathrm{inh}}\equiv\tau N=\Lambda\frac{\hbar^2|\omega_{\min}|}{2|V_{\max}|^2}. \quad (38)$$

We call this duration the inhibition time (it is natural to call this duration the Zeno time, but this term already has a different meaning).

The full probability of the jump from level $|i\alpha\rangle$ to other levels is $W(i\alpha)=\Sigma_{f,\alpha_1}W(i\alpha\rightarrow f\alpha_1)$. From Eq. (37), we obtain

$$W(i\alpha)=\frac{2}{\hbar^2\Lambda}\sum_{f,\alpha_1}\frac{1}{|\omega_{if}|}\int_0^\tau dt|V(t+t_0)_{f\alpha_1,i\alpha}|^2. \quad (39)$$

If the matrix elements of the perturbation V between different levels are of the same size, then the jump probability increases linearly with the number of energy levels. This behavior has been observed in Ref. [23].

Due to the unitarity of the operator $\hat{U}(t,\xi)$, it follows from Eq. (9) that the superoperator $S(\tau,t_0)$ obeys the equalities

$$\sum_{p,\alpha} S(\tau,t_0)^{n\alpha_1,m\alpha_2}_{p\alpha,p\alpha}=\delta_{nm}\delta_{\alpha_1,\alpha_2}, \tag{40a}$$

$$\sum_{n,\alpha} S(\tau,t_0)^{n\alpha,n\alpha}_{p\alpha_1,r\alpha_2}=\delta_{pr}\delta_{\alpha_1,\alpha_2}. \tag{40b}$$

If the system has a finite number of energy levels, the density matrix of the system is diagonal, and all states are equally occupied [i.e., $\rho(t_0)_{n\alpha_1,m\alpha_2}=(1/K)\delta_{nm}\delta_{\alpha_1,\alpha_2}$, where K is the number of the energy levels], then from Eq. (40b) it follows that $S(\tau,t_0)\rho(t_0)=\rho(t_0)$. Such a density matrix is the stable point of the map $\rho\rightarrow S\rho$. Therefore, we can expect that after a large number of measurements, the density matrix of the system tends to this density matrix.

When Λ is large and the duration of the measurement is small, we can neglect the nondiagonal elements in the density matrix of the system, since they are always of order Λ^{-1}. Replacing $F(\nu)$ by $2C\delta(\nu)$ in Eqs. (A1), (A2), and (A3) and neglecting the elements of the superoperator S that cause the arising of the nondiagonal elements of the density matrix, we can write the equation for the superoperator S as

$$S(\tau,t_0)^{n\alpha_1,m\alpha_2}_{p\alpha_3,r\alpha_4}\approx\delta_{pn}\delta(\alpha_3,\alpha_1)\delta_{rm}\delta(\alpha_4,\alpha_2)\delta_{pr}+\frac{1}{\Lambda}A(\tau,t_0)^{n,\alpha_1,\alpha_2}_{p,\alpha_3,\alpha_4}\delta_{pr}\delta_{nm}, \tag{41}$$

where

$$\begin{aligned}A(\tau,t_0)^{n,\alpha_1,\alpha_2}_{p,\alpha_3,\alpha_4}=&\frac{2}{\hbar^2|\omega_{np}|}\int_0^\tau dt\,V(t+t_0)_{p\alpha_3,n\alpha_1}\\&\times V(t+t_0)_{n\alpha_2,p\alpha_4}-\delta_{pn}\delta(\alpha_4,\alpha_2)\\&\times\sum_{s,\alpha}\frac{1}{\hbar^2|\omega_{sn}|}\int_0^\tau dt\,V(t+t_0)_{n\alpha_3,s\alpha}\\&\times V(t+t_0)_{s\alpha,n\alpha_1}-\delta_{pn}\delta(\alpha_3,\alpha_1)\\&\times\sum_{s,\alpha}\frac{1}{\hbar^2|\omega_{ns}|}\int_0^\tau dt V(t+t_0)_{s\alpha,n\alpha_4}\\&\times V(t+t_0)_{n\alpha_2,s\alpha}.\end{aligned} \tag{42}$$

Then for the diagonal elements of the density matrix, we have $\rho(\tau+t_0)\approx\rho(t_0)+(1/\Lambda)A(\tau,t_0)\rho(t_0)$, or

$$\frac{d}{dt}\hat{\rho}(t)\approx\frac{1}{\Lambda\tau}A(\tau,t)\hat{\rho}(t). \tag{43}$$

If the perturbation V does not depend on t, then it follows from Eq. (43) that the diagonal elements of the density matrix evolve exponentially.

A. Example

As an example, we will consider the evolution of the measured two-level system. The system is forced by the perturbation V, which induces the jumps from one state to another. The Hamiltonian of this system is

$$\hat{H}=\hat{H}_0+\hat{V}, \tag{44}$$

where

$$\hat{H}_0=\frac{\hbar\omega}{2}\hat{\sigma}_3, \tag{45}$$

$$\hat{V}=v\hat{\sigma}_+ + v^*\hat{\sigma}_- . \tag{46}$$

Here $\sigma_1,\sigma_2,\sigma_3$ are Pauli matrices and $\sigma_\pm=\frac{1}{2}(\sigma_1\pm i\sigma_2)$. The Hamiltonian $\hat{H}_0$ has two eigenfunctions $|0\rangle$ and $|1\rangle$ with the eigenvalues $-\hbar(\omega/2)$ and $\hbar(\omega/2)$, respectively. The evolution operator of the unmeasured system is

$$\hat{U}(t)=\cos\left(\frac{\Omega}{2}t\right)-\frac{2i}{\hbar\Omega}\hat{H}\,\sin\left(\frac{\Omega}{2}t\right), \tag{47}$$

where

$$\Omega=\sqrt{\omega^2+4\frac{|v|^2}{\hbar^2}}. \tag{48}$$

If the initial density matrix is $\rho(0)=|1\rangle\langle 1|$, then the evolution of the diagonal elements of the unmeasured system's density matrix is given by the equations

$$\rho_{11}(t)=\cos^2\left(\frac{\Omega}{2}t\right)+\left(\frac{\omega}{\Omega}\right)^2\sin^2\left(\frac{\Omega}{2}t\right), \tag{49a}$$

$$\rho_{00}(t)=\left[1-\left(\frac{\omega}{\Omega}\right)^2\right]\sin^2\left(\frac{\Omega}{2}t\right). \tag{49b}$$

Let us consider now the dynamics of the measured system. The equations for the diagonal elements of the density matrix [Eq. (43)] for the system under consideration are

$$\frac{d}{dt}\rho_{11}\approx-\frac{1}{t_{\rm inh}}(\rho_{11}-\rho_{00}), \tag{50a}$$

$$\frac{d}{dt}\rho_{00}\approx-\frac{1}{t_{\rm inh}}(\rho_{00}-\rho_{11}), \tag{50b}$$

where the inhibition time, according to Eq. (38), is

$$t_{\rm inh}=\frac{\Lambda}{2\omega}\left|\frac{\hbar\omega}{v}\right|^2. \tag{51}$$

The solution of Eqs. (50) with the initial condition $\rho(0)=|1\rangle\langle 1|$ is

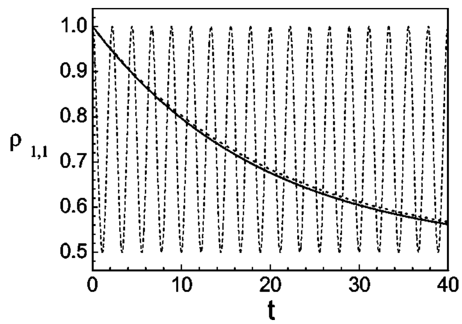

FIG. 1. The occupation of the initial level 1 of the measured two-level system calculated according to Eqs. (5), (9), (47), and (54). The used parameters are $\hbar=1$, $\sigma^2=1$, $\omega=2$, $v=1$. In this system of units, time is dimensionless. The strength of the measurement $\lambda=50$ and the duration of the measurement $\tau=0.1$. The exponential approximation (52a) is shown as a dotted line. For comparison, the occupation of the level 1 of the unmeasured system is also shown (dashed line).

$$\rho_{11}(t)=\frac{1}{2}\left[1+\exp\left(-\frac{2}{t_{\text{inh}}}t\right)\right], \tag{52a}$$

$$\rho_{00}(t)=\frac{1}{2}\left[1-\exp\left(-\frac{2}{t_{\text{inh}}}t\right)\right]. \tag{52b}$$

From Eq. (40b), it follows that if the density matrix of the system is

$$\hat{\rho}_f=\frac{1}{2}(|0\rangle\langle 0|+|1\rangle\langle 1|), \tag{53}$$

then $S(\tau)\hat{\rho}_f=\hat{\rho}_f$. Hence, when the number of measurements tends to infinity, the density matrix of the system approaches $\hat{\rho}_f$.

We have performed the numerical analysis of the dynamics of the measured two-level system (44)–(46) using Eqs. (5), (9), and (47) with the Gaussian correlation function (15),

$$F(\nu)=\exp\left(-\frac{\nu^2}{2\sigma^2}\right). \tag{54}$$

From the condition $\int_{-\infty}^{\infty}F(\nu)d\nu=2C$, we have $C=\sigma\sqrt{\pi/2}$. The initial state of the system is $|1\rangle$. The matrix elements of the density matrix $\rho(t)_{11}$ and $\rho(t)_{10}$ are represented in Fig. 1 and Fig. 2, respectively. In Fig. 1, the approximation (52a) is also shown. This approach is close to the exact evolution. The matrix element $\rho(t)_{11}$ for two different values of λ is shown in Fig. 3. We see that for larger λ, the evolution of the system is slower.

The influence of the repeated nonideal measurements on the two-level system driven by the periodic perturbation has also been considered in Refs. [23–26]. Similar results have been found: the occupation of the energy levels changes exponentially with time, approaching the limit $\frac{1}{2}$.

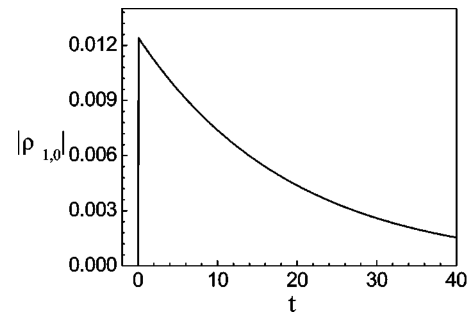

FIG. 2. The nondiagonal element of the density matrix of the measured two-level system. Used parameters are the same as in Fig. 1.

VI. DECAYING SYSTEM

We consider a system that consists of two parts. We can treat the first part as an atom and the second part as the field (reservoir). The energy spectrum of the atom is discrete and the spectrum of the field is continuous. The Hamiltonians of these parts are $\hat{H}_0$ and $\hat{H}_1$, respectively, and the eigenfunctions are $|n\rangle$ and $|\alpha\rangle$,

$$\hat{H}_0|n\rangle=E_n|n\rangle, \tag{55a}$$

$$\hat{H}_1|\alpha\rangle=E_\alpha|\alpha\rangle. \tag{55b}$$

There is the interaction between the atom and the field represented by the operator $\hat{V}$. So the Hamiltonian of the system is

$$\hat{H}_S=\hat{H}_0+\hat{H}_1+\hat{V}. \tag{56}$$

The basis for the full system is $|n\alpha\rangle=|n\rangle\otimes|\alpha\rangle$.

When the measurement is not performed, such a system exhibits exponential decay, valid for the intermediate times. The decay rate is given according to Fermi's Golden Rule,

$$R(i\alpha_1\to f\alpha_2)=\frac{2\pi}{\hbar}|V_{f\alpha_2,i\alpha_1}|^2\rho(\hbar\omega_{if}), \tag{57}$$

where

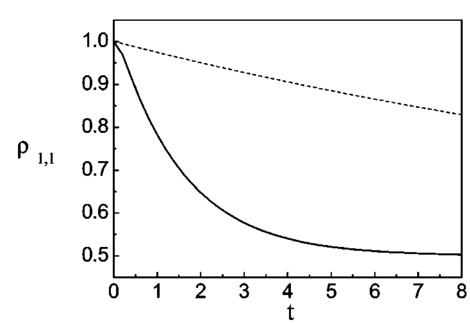

FIG. 3. The occupation of the initial level 1 of the measured two-level system for different strengths of the measurement: $\lambda=50$, $\tau=0.1$ (dashed line) and $\lambda=5$, $\tau=0.2$ (solid line). Other parameters are the same as in Fig. 1.

$$\frac{1}{\hbar}(E_{\alpha_2}-E_{\alpha_1})=\omega_{if} \tag{58}$$

and $\rho(E)$ is the density of the reservoir's states.

When the energy level of the atom is measured, we can use the perturbation theory, as it is in the discrete case. The initial state of the field is a vacuum state $|0\rangle$ with energy $E_0=0$. Then the density matrix of the atom is $\hat{\rho}_0(\tau)=\mathrm{Tr}_1\{\hat{\rho}(\tau)\}=\mathrm{Tr}_1\{S(\tau)\hat{\rho}(0)\}$ or $\hat{\rho}_0(\tau)=S_{\mathrm{ef}}(\tau)\hat{\rho}_0(0)$, where S_{ef} is an effective superoperator,

$$S_{\mathrm{ef}}(\tau)^{nm}_{pr}=\sum_{\alpha} S(\tau)^{n0,m0}_{p\alpha,r\alpha}. \tag{59}$$

When the states of the atom are weakly coupled to a broadband of states (continuum), the transitions back to the excited state of the atom can be neglected (i.e., we neglect the influence of emitted photons on the atom). Therefore, we can use the superoperator S_{ef} for a determination of the evolution of the atom.

Since the states in the reservoir are very dense, one can replace the sum over α by an integral over E_α,

$$\sum_{\alpha} \cdots=\int dE_\alpha \rho(E_\alpha)\cdots,$$

where $\rho(E_\alpha)$ is the density of the states in the reservoir.

A. The spectrum

The density matrix of the field is $\hat{\rho}_1(\tau)=\mathrm{Tr}_0\{\hat{\rho}(\tau)\}=\mathrm{Tr}_0\{S(\tau)\hat{\rho}(0)\}$. The diagonal elements of the field's density matrix give the spectrum. If the initial state of the atom is $|i\rangle$, then the distribution of the field's energy is $W(E_\alpha)=\rho_1(\tau)_{\alpha\alpha}=\Sigma_f S(\tau)^{i0,i0}_{f\alpha,f\alpha}$. From Eqs. (A1), (A2), and (A3), we obtain

$$W(E_\alpha)=\sum_f \frac{2\pi}{\hbar^2}|V_{f\alpha,i0}|^2\tau P\left(\frac{E_\alpha}{\hbar}\right)_{if}, \tag{60}$$

where $P(\omega)_{if}$ is given by Eq. (34). From Eq. (60), we see that $P(\omega)$ is the measurement-modified shape of the spectral line.

The integral in Eq. (34) is small when the exponent oscillates more rapidly than the function F. This condition is fulfilled when $(E/\hbar)-\omega_{if}\gtrsim\lambda\omega_{if}/C$. Consequently, the width of the spectral line is

$$\Delta E_{if}=\Lambda\hbar\omega_{if}. \tag{61}$$

The width of the spectral line is proportional to the strength of the measurement (this equation is obtained using the assumption that the strength of the interaction with the measuring device λ is large and, therefore, the natural width of the spectral line can be neglected). The broadening of the spectrum of the measured system is also reported in Ref. [12] for the case of an electron tunneling out of a quantum dot.

B. The decay rate

The probability of the jump from the state i to the state f is $W(i\rightarrow f;\tau)=S_{\mathrm{ef}}(\tau)^{ii}_{ff}$. From Eq. (59), it follows that

$$W(i\rightarrow f;\tau)=\sum_{\alpha} W(i0,\rightarrow f\alpha,\tau). \tag{62}$$

Using Eq. (33), we obtain the equality

$$W(i\rightarrow f;\tau)=\frac{2\pi\tau}{\hbar^2}\int_{-\infty}^{+\infty}d\omega\, G(\omega)_{fi}P(\omega)_{if}, \tag{63}$$

where

$$G(\omega)_{fi}=\int dE_\alpha\rho(E_\alpha)G(\omega)_{f\alpha,i0}. \tag{64}$$

The expression for $G(\omega)$ according to Eq. (36) is

$$G(\omega)_{fi}=\hbar\rho(\hbar\omega)|V_{fE_\alpha=\hbar\omega,i0}|^2. \tag{65}$$

The quantity $G(\omega)$ is the reservoir coupling spectrum.

The measurement-modified decay rate is $R(i\rightarrow f)=(1/\tau)W(i\rightarrow f;\tau)$. From Eq. (63), we have

$$R(i\rightarrow f)=\frac{2\pi}{\hbar^2}\int_{-\infty}^{\infty}d\omega\, G(\omega)_{fi}P(\omega)_{if}. \tag{66}$$

Equation (66) represents a universal result: the decay rate of the frequently measured decaying system is determined by the overlap of the reservoir coupling spectrum and the measurement-modified level width. This equation was derived by Kofman and Kurizki [14], assuming the ideal instantaneous projections. We show that Eq. (66) is valid for the more realistic model of the measurement as well. An equation similar to Eq. (66) has been obtained in Ref. [27], considering a destruction of the final decay state.

Depending on the reservoir spectrum $G(\omega)$ and the strength of the measurement, the inhibition or acceleration of the decay can be obtained. If the interaction with the measuring device is weak and, consequently, the width of the spectral line is much smaller than the width of the reservoir spectrum, then the decay rate equals the decay rate of the unmeasured system, given by Fermi's Golden Rule (57). In the intermediate region, when the width of the spectral line is rather small compared with the distance between ω_{if} and the nearest maximum in the reservoir spectrum, the decay rate grows with an increase of Λ. This results in the anti-Zeno effect.

If the width of the spectral line is much greater compared both with the width of the reservoir spectrum and the distance between $\omega_{i,f}$ and the centrum of the reservoir spectrum, then the decay rate decreases when Λ increases. This results in the quantum Zeno effect. In such a case, we can use the approximation

$$G(\omega)_{fi}\approx\hbar B_{fi}\delta(\omega-\omega_R), \tag{67}$$

where B_{fi} is defined by the equality $B_{fi}=(1/\hbar)\int G(\omega)_{fi}d\omega$ and ω_R is the centrum of $G(\omega)$. Then from Eq. (66) we obtain the decay rate $R(i\rightarrow f)\approx(2\pi/\hbar)B_{fi}P(\omega_{if})_{if}$. From Eq. (34), using the condition $\Lambda\tau|\omega_{if}|\gg 1$ and the equality $\int_{-\infty}^{\infty}F(\nu)d\nu=2C$, we obtain

$$P(\omega_{if})_{if}=\frac{1}{\pi\Lambda\omega_{if}}. \tag{68}$$

Therefore, the decay rate is equal to

$$R(i\rightarrow f)\approx\frac{2B_{fi}}{\Lambda\hbar\omega_{if}}. \tag{69}$$

The obtained decay rate is insensitive to the spectral shape of the reservoir and is inversely proportional to the measurement strength Λ.

VII. SUMMARY

In this work, we investigate the quantum Zeno effect using the definite model of the measurement. We take into account the finite duration and the finite accuracy of the measurement. The general equation for the probability of the jump during the measurement is derived [Eq. (33)]. The behavior of the system under the repeated measurements depends on the strength of measurement and on the properties of the system.

When the strength of the interaction with the measuring device is sufficiently large, the frequent measurements of the system with a discrete spectrum slow down the evolution. However, the evolution cannot be fully stopped. Under the repeated measurements, the occupation of the energy levels changes exponentially with time, approaching the limit of the equal occupation of the levels. The jump probability is inversely proportional to the strength of the interaction with the measuring device.

In the case of a continuous spectrum, the measurements can cause inhibition or acceleration of the evolution. Our model of the continuous measurement gives the same result as the approach based on the projection postulate [14]. The decay rate is equal to the convolution of the reservoir coupling spectrum with the measurement-modified shape of the spectral line. The width of the spectral line is proportional to the strength of the interaction with the measuring device. When this width is much greater than the width of the reservoir, the quantum Zeno effect takes place. Under these conditions, the decay rate is inversely proportional to the strength of the interaction with the measuring device. In a number of decaying systems, however, the reservoir spectrum $G(\omega)$ grows with frequency almost up to the relativistic cutoff, and the strength of the interaction required for the appearance of the quantum Zeno effect is so high that the initial system is significantly modified. When the spectral line is not very broad, the decay rate may be increased by the measurements more often than it may be decreased and the quantum anti-Zeno effect can be obtained.

APPENDIX: THE SUPEROPERATOR

We obtain the superoperator S in the second-order approximation substituting the approximate expression for the evolution operator (28) into Eq. (9). Thus we have

$$S(\tau,t_0)=S^{(0)}(\tau)+S^{(1)}(\tau,t_0)+S^{(2)}(\tau,t_0), \tag{A1}$$

where $S^{(0)}(\tau)$ is the superoperator of the unperturbed measurement given by Eq. (16), $S^{(1)}(\tau,t_0)$ is the first-order correction,

$$\begin{aligned}S^{(1)}(\tau,t_0)_{p\alpha_3,r\alpha_4}^{n\alpha_1,m\alpha_2}=&\frac{1}{i\hbar}\delta_{rm}\delta(\alpha_4,\alpha_2)\exp(i\omega_{r\alpha_4,p\alpha_3}\tau)\\&\times\int_0^{\tau}dt\,V(t+t_0)_{p\alpha_3,n\alpha_1}\\&\times\exp(i\omega_{p\alpha_3,n\alpha_1}t)F(\lambda(\omega_{rp}\tau+\omega_{pn}t))\\&-\frac{1}{i\hbar}\delta_{pn}\delta(\alpha_3,\alpha_1)\exp(i\omega_{r\alpha_4,p\alpha_3}\tau)\\&\times\int_0^{\tau}dt\,V(t+t_0)_{m\alpha_2,r\alpha_4}\\&\times\exp(i\omega_{m\alpha_2,r\alpha_4}t)F(\lambda(\omega_{rp}\tau+\omega_{mr}t)),\end{aligned} \tag{A2}$$

and $S^{(2)}(\tau,t_0)$ is the second-order correction,

$$\begin{aligned}S^{(2)}(\tau,t_0)_{p\alpha_3,r\alpha_4}^{n\alpha_1,m\alpha_2}=&\frac{1}{\hbar^2}\exp(i\omega_{r\alpha_4,p\alpha_3}\tau)\int_0^{\tau}dt_1\int_0^{\tau}dt_2V(t_1+t_0)_{p\alpha_3,n\alpha_1}V(t_2+t_0)_{m\alpha_2,r\alpha_4}F(\lambda(\omega_{rp}\tau+\omega_{pn}t_1+\omega_{mr}t_2))\\&\times\exp(i\omega_{p\alpha_3,n\alpha_1}t_1+i\omega_{m\alpha_2,r\alpha_4}t_2)-\frac{1}{\hbar^2}\delta_{rm}\delta(\alpha_4,\alpha_2)\exp(i\omega_{r\alpha_4,p\alpha_3}\tau)\\&\times\sum_{s,\alpha}\int_0^{\tau}dt_1\int_0^{t_1}dt_2V(t_1+t_0)_{p\alpha_3,s\alpha}V(t_2+t_0)_{s\alpha,n\alpha_1}F(\lambda(\omega_{rp}\tau+\omega_{ps}t_1+\omega_{sn}t_2))\\&\times\exp(i\omega_{p\alpha_3,s\alpha}t_1+i\omega_{s\alpha,n\alpha_1}t_2)-\frac{1}{\hbar^2}\delta_{pn}\delta(\alpha_3,\alpha_1)\exp(i\omega_{r\alpha_4,p\alpha_3}\tau)\sum_{s,\alpha}\int_0^{\tau}dt_1\int_0^{t_1}dt_2V(t_2+t_0)_{m\alpha_2,s\alpha}\\&\times V(t_1+t_0)_{s\alpha,r\alpha_4}F(\lambda(\omega_{rp}\tau+\omega_{sr}t_1+\omega_{ms}t_2))\exp(i\omega_{s\alpha,r\alpha_4}t_1+i\omega_{m\alpha_2,s\alpha}t_2),\end{aligned} \tag{A3}$$

where

$$\omega_{n\alpha_1,m\alpha_2}=\omega_{nm}+\frac{E_1(n,\alpha_1)-E_1(m,\alpha_2)}{\hbar}. \quad \text{(A4)}$$

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