Supplementary Material

In the Supplementary Material, we present the derivation of the cavity-mediated long-range interaction and the self-consistent atomic density distributions.

I. LATTICE MODEL WITH EFFECTIVE GLOBAL INTERACTION.

Let us consider an atom affected by a one-dimensional static bichromatic incommensurate potential along the x-axis. The atom is also coupled to a single-mode cavity field propagating along the same x-axis. The single-particle Hamiltonian for such an atom is given by

$$\begin{aligned} H &= -\frac{\hbar^2}{2m}\partial_x^2 + V_1\cos^2(k_1 x) + V_2\cos^2(k_2 x+\phi) \\ &+ \hbar\eta(\hat{a}^\dagger+\hat{a})\cos(k_c x+\phi_c) + U\cos^2(k_c x+\phi_c)\hat{a}^\dagger\hat{a}\,, \end{aligned} \tag{S1}$$

where $\eta = \eta_0\cos(k_p z_0)$ describes the transverse (stranding-wave) pumping via atoms, $\eta_0 = g\Omega/\Delta_a$ is the strength of the interference between the pumping laser and the cavity field, $U = g^2/\Delta_a$ is the dispersive coupling strength, $\Delta_a = \omega_p - \omega_a$ denotes the detuning between the pumping laser ω_p and atomic transition frequency ω_a, Ω is the strength (Rabi frequency) of the pumping laser, and g is the single-photon Rabi frequency of the cavity mode. Here also ϕ (ϕ_c) is the relative phase between the second incommensurate lattice (cavity-assisted potential) and the primary lattice potential.

One can write the many-body Hamiltonian of the system in the framework of the second quantization as

$$\hat{\mathcal{H}} = \int dx \hat{\psi}^\dagger(x) H \hat{\psi}(x) - \hbar\Delta_c\hat{a}^\dagger\hat{a}, \tag{S2}$$

where $\Delta_c = \omega_p - \omega_c$ denotes the detuning between the pumping laser frequency ω_p and cavity mode frequency ω_c, $\hat{\psi}(x)$ ($\hat{\psi}^\dagger(x)$) are the operators for the annihilation (creation) of an atom at a position x satisfying the bosonic commutation relations. We assume that the primary lattice potential V_1 is much larger than the secondary incommensurate lattice potential V_2, as well as the cavity-assisted dynamical potential. This is well justified for small photon numbers in the vicinity of the superradiance transition discussed in this paper. We expand the operator $\hat{\psi}(x)$ in the basis of the Wannier functions of the lowest band of the static primary lattice:

$$\hat{\psi}^\dagger(x) = \sum_j w_j^*(x)\hat{c}_j^\dagger, \qquad \hat{\psi}(x) = \sum_j w_j(x)\hat{c}_j, \tag{S3}$$

where $\hat{c}_j$ ($\hat{c}_j^\dagger$) denotes the annihilation (creation) operator of a boson at a site j, and $w_j(x)$ is the Wannier function localized on the j-th site. Substituting Eq. (S3) into Eq. (S2), the total Hamiltonian takes the form

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_{\mathrm{AA/GAA}} + \eta(\hat{a}^\dagger+\hat{a})\sum_j\cos(2\pi\gamma_c j)\hat{c}_j^\dagger\hat{c}_j - [\Delta_c - U\sum_j\cos^2(2\pi\gamma_c j)\hat{c}_j^\dagger\hat{c}_j]\hat{a}^\dagger\hat{a}, \tag{S4}$$

with $\gamma_c = k_c/(2k_1)$, where $\hat{\mathcal{H}}_{\mathrm{AA/GAA}}$ is given by Eq. (1) in the main text and we have chosen $\phi = \phi_c = 0$ without loss of generality.

In the steady state, the cavity field acquires

$$\hat{a} = \frac{\eta}{\tilde{\Delta}_c + i\kappa}\sum_j\cos(2\pi\gamma_c j)\hat{n}_j, \tag{S5}$$

where $\tilde{\Delta}_c = \Delta_c - U\sum_j\cos^2(2\pi\gamma_c j)\hat{n}_j$ and $\hat{n}_j = \hat{c}_j^\dagger\hat{c}_j$. Inserting Eq. (S5) into Eq. (S4), we arrive at the Hamiltonian $\hat{\mathcal{H}} = \hat{\mathcal{H}}_{\mathrm{AA/GAA}} + \hat{\mathcal{H}}_{\mathrm{global}}$ containing the cavity-mediated long-range interaction

$$\hat{\mathcal{H}}_{\mathrm{global}} = \sum_{jj'} U_{jj'}\hat{n}_j\hat{n}_{j'}. \tag{S6}$$

Here $U_{jj'} = U_1 \cos(2\pi\gamma_c j)\cos(2\pi\gamma_c j')$ describes an effective interaction between atoms at the sites j and j', with $U_1 \equiv \frac{\eta^2\tilde{\Delta}_c}{\tilde{\Delta}_c^2+\kappa^2}$ being the strength of the interaction.

With the Gauss approximation of the Wannier state, $J \sim (V_1/E_R)^{0.75}\exp(-\sqrt{V_1/E_R})$ and $\chi \sim (V_2/E_R)\exp(-1/\sqrt{V_1/E_R})$ with the recoil energy of the prime lattice $E_R = \hbar^2 k_1^2/(2m)$ [50], one can tune χ and J by varying V_2/V_1.

II. SELF-CONSISTENT ATOMIC DENSITY DISTRIBUTIONS

In Fig. S1 we plot the atomic density distributions versus the disorder strength which increases from zero (a) to the finite values (b,c,d) reaching the localization regime (e). We can see that increasing the disorder strength can drive the extended states for $\eta < \eta_c$ to the localized ones for $\eta > \eta_c$ (Fig. S1c). Especially, when the disorder strength tuned across the delocalization-localization point, the photon scattering always induces a transition from the extended normal phase to a superradiance phase with well localized state (Fig. S1d). In particular, when the system is in the localization phase and the superradiant phase with finite $\eta > \eta_c = 0$, the states become even more localized compared with the case without cavity field (Fig. S1e). This implies that in the mean-field level the weak cavity field even strengthens the localization. As beyond the mean-field treatment the cavity field induces the long-rang interaction (the former section), the many-body localization could be expected in the reasonable regimes. This analysis is consistent with the very recent advances. A parameter space for the many-body localization was recently reported for the cavity-mediated all-to-all coupling [1] and for a similar model of the globally Floquet driven system [2].

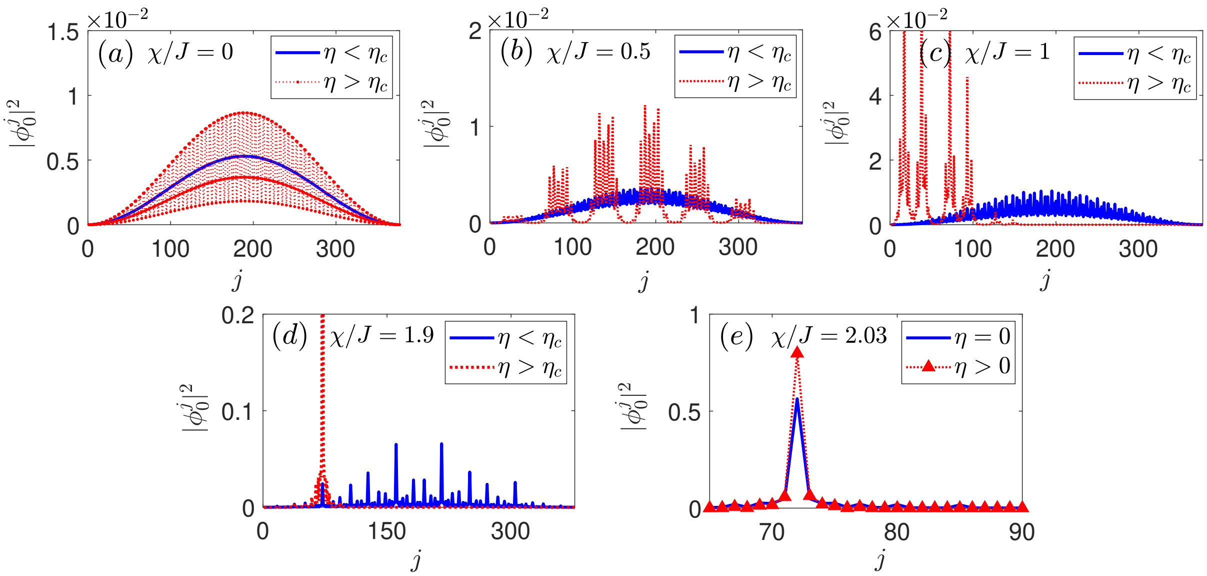

FIG. S1: The atomic density distributions for different disorder strength χ/J. Here, the parameters are $L = 377$, $N = 100$, $\Delta_c/J = -1$ and $\kappa/J = 1$.

[1] P. Sierant, K. Biedroń, G. Morigi, and J. Zakrzewski, SciPost Phys. **7**, 008 (2019).
[2] N. Ng and M. Kolodrubetz, Phys. Rev. lett. **122**, 240402 (2019).